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WALMSLEY'S
INTRODUCTION TO GEOMETRY.

EUCLID BOOK 1.





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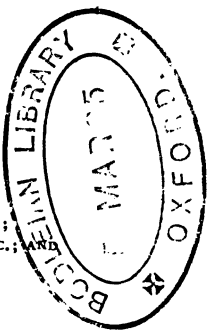
FOR THE USE OF BEGINNERS.

CONSISTING OF

EUCLID'S ELEMENTS,

BOOK I.

ACCOMPANIED BY NUMEROUS EXPLANATIONS,
QUESTIONS, AND EXERCISES.



BY

JOHN WALMSLEY, B.A.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY;
AUTHOR OF "PLANE TRIGONOMETRY AND LOGARITHMS," &C.; AND
HEAD MASTER OF THE GRAMMAR SCHOOL, ECCLES.

New Edition, revised, corrected, and improved.

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1884.

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NOTE.—The Diagrams at the end are upon tough paper, and each of them opens out for ready use with any other page of the book.

PREFACE TO SECOND EDITION.

THE principles of this little work have met with much approbation from teachers and authorities of high standing. In the preparation of a new edition, therefore, a great effort has been made to carry out those principles fully, while providing for as small an expenditure of time and labour on the part of both pupil and teacher as will be effective. The book is now stereotyped, and hence teachers may use it with the assurance that it will meet with little or no alteration in future.

In reference to the changes which have been made, it may be noted that the extracts from the London University Matriculation papers have apparently disappeared. All the problems in them, however, which were new or otherwise important, have been absorbed into the "Papers for Revision." Room was made for them by omitting some less useful questions, and by adding one new paper. In place of the portions of Matriculation papers referred to, those of the Science and Art Department have been brought up to date, and now form a most useful series of ten. In the body of the work, questions formerly occurred here and there which were found to be more or less vague, and therefore to invite guessing answers. Some of these have been amended, and others removed. Other questions, again, involved repetition carried somewhat to excess, and for that reason have been considered unnecessary.

The bulk of the work is still what it was; and the differences, which will be at once evident on comparison with the former edition, are mainly due to re-arrangement of matter. This has been carried out in accordance with the suggestions of experience, with a view to render the work more practically useful as a class-book. We may add here that anyone who will examine the papers recently set in the London University Matriculation Examinations, or those of the Science and Art Department, will perceive that for these examinations some such training as is afforded in this book is fast becoming not only desirable, but indispensable. It appears also to be becoming more and more so in all other examinations of good reputation.

J. W.

August, 1884.

FROM PREFACE TO THE FIRST EDITION.

At present, it is not rash to assume that, out of the vast number of our youths who are trained in Euclid, but a low percentage ever become Geometers in any true sense of the word. In the circumstances of modern times, this must be admitted to be a fair subject of serious reproach; whatever else may be accomplished by Euclidian training as it is. But, in the view of the present writer, there is no need to dispense with Euclid on this account; for his text may be thoroughly learnt for its own peculiar benefits, not only as well but better, when it is accompanied by all that original work which will make a pupil into more or less of a true Geometer.

Among the most distinctive features of this work are the *number* and the *position* of the explanations, questions, and exercises. But the *nature* of all these, which will not be so readily obvious, is intended to be the most useful of its characteristics.

The earlier ones are not only designed to make clear the meanings of the definitions, &c., but also to take advantage of these to gradually develop the power of constructing simple geometrical arguments; so that, on reaching the propositions, the pupil will be sure to enter upon them with intelligent interest and sufficient geometrical power. He will then never be found among those who go far through the propositions with no notion of distinguishing their various parts, except by differences of type or from direct labelling; and who

know they are at the end of a proposition only by meeting with the well-known sign-post of the three capital letters.

In the further exercises which accompany the propositions, every proposition, except some of the very easiest of the later ones, meets with analytical consideration before it is entered upon. By means of a series of questions and easy exercises, the pupil is led to study each part of the proposition in turn, and then himself to build the parts together. At the same time, this is not done directly, so as to produce the exact form of the coming proposition, but with diverse features and conditions, so as to afford variety and scope for originality and mental exercise.

The value of such exercises will perhaps be admitted by all. But, in giving so much original work, it may be objected that no account is taken of the time at the teacher's disposal; and that it can never be got through. In this objection a double misconception would be involved.

A large portion of the work is set so that it shall admit of explicit answering and *ready correction*. For pupils who are not very young, many portions of the exercises may be taken as simply *oral* practice in class. There are numerous other questions and exercises whose answers are determinate and brief; and the easy deductions which often occur may be added to them by simply requiring in each case the construction and the quoting of those propositions on which the proof will be founded, full proofs being afterwards taken orally with the blackboard. Proceeding in this way, it is evident that in regard to much of the practical work of the class the teacher may correct, with or without the aid of the printed "Key," very much as he would in the case of exercises in Arithmetic or Latin. It is recommended that the most important of such examples as the teacher finds it needful to discuss with the aid of the blackboard, should be set for writing out in full with the next exercise. To glance

over portions of a coming exercise also with the aid of the blackboard, will frequently be found of advantage with younger pupils. Pursuing the course here indicated, the expenditure of time required for the thorough study of these pages is probably far less than would be estimated.

On the other hand, it is quite possible to under-estimate the time required to arrive at an equal result by the usual means. By the method described above, a natural and *continuous* training may be afforded to pupils taken in classes; which is more than can be said for systems which involve the use of the ordinary sets of deductions, commonly and properly enough styled mere "collections." To succeed with these, teaching power of a high quality is indispensable; and it must be very liberally expended. Indeed, a high authority on this subject, the head-master of Clifton College, has asserted that boys cannot be got to work systematically upon "deductions."

The text of Simson is given throughout with a clear arrangement and in conspicuous type, so as to be easily followed when specially revising it. A few changes have been made in words and phraseology, for purposes of emendation or greater clearness; but this has been done with great care. By the use of Italics it has been sought to emphasize those statements whose importance learners are most apt to underestimate. The Italics are thus used with their ordinary application; and it is simply incidental that they frequently assist in the logical division of the propositions. The diagrams will be found good and clear, and are repeated wherever the text requires it.

For those who are favourable to replacing Euclid's proofs in some cases, a series of alternative ones is given in an Appendix, consisting of those which are likely to find most favour with teachers. They are expressed with the aid of symbols, in order to assist, at the same time, such as may

desire to initiate their pupils early into the use of symbols. Those symbols and abbreviations only are used which would not be objected to in any of the ordinary examinations.

For some pupils there may be superabundance in the materials herein provided; but in no case can this be a serious fault; for to supply the omissions of a scanty treatise is by no means so easy as to pass by portions not needed in a fuller one.

The book is intended to be suitable for preceding the study of any good edition of Euclid.

J. W.

EUCLID—BOOK I.

INTRODUCTION.

THE name Geometry is derived from two Greek words—*ge*, the earth, and *metron*, a measure. The science of Geometry appears to have taken its rise from the art of measuring land, most probably in Egypt, and long before the time of Jesus Christ. The ancient Greeks cultivated the subject, and made great progress in increasing human knowledge of it. About 300 B.C., a famous Greek named Euclid, who lived at Alexandria in Egypt, wrote a treatise on the "Elements of Geometry." This treatise has been one of the most wonderful books ever written. Its interesting history cannot be told here; but we require to mention that the present little book is founded upon a portion of Euclid's work, and is intended to enable the beginner in Geometry to understand and learn thoroughly what that portion contains. What is here called learning will not be considered complete until it has become easy to put the knowledge acquired to practical use.

The portion of this book which is due, in the main, to Euclid, will be found printed in larger type.

DEFINITIONS.

1. WHAT ARE DEFINITIONS?

"*By a definition we mean a precise statement of the qualities which are just sufficient to mark out a class, and thus to tell exactly what things belong to a class and what do not.*"—(JEVONS.)

Notice the word 'just' in this explanation, and remember that a thing

may have other qualities besides the one or more which we use in defining its name; but any definition which gives more of those qualities than are necessary for its own purposes, is not a strictly proper definition.

We may usefully notice two sorts of improper definitions.

A *redundant definition* of a class of things is a definition which states *more* qualities than are sufficient to mark out that class from all other things.

A definition of a class of things is *deficient* when it does not state such qualities as are sufficient to distinguish that class from all other things.

Almost all the definitions which will be given hereafter were written by Euclid nearly 2,200 years ago. They are very good, taken as a whole; but, like all other human work, they have their faults also, here and there. Some of these we will try to find out, and notice as we go on.

The word 'definition' is derived from the Latin *definio*, I limit.

2. Points.

The learner may here be reminded that Euclid's definitions will be in larger print. He should study them, as they come before him, until he can repeat them from memory *fluently*. In order to give him, for the present at least, the opportunity of doing a little at a time and doing it well, these early articles will each embrace a very narrow, although of course increasing, range.

1. A point is that which has no parts, or which has no magnitude.

All such points as a dot on a blackboard, or a full stop on paper, *have* magnitude or size, however small they may be; they have both length and breadth, and even thickness or depth; and can therefore be divided into parts. Yet in these cases the magnitude is generally an accidental quality, which we do not want *for its own sake*, but rather that we may *see* the point, and have something to guide the eye in fixing the position. The *position* or place, which is marked by the point, is what we usually wish to consider. In Geometry it is *always* so; hence some writers prefer to say on this subject,—"A point is that which has position but no parts."

A point, as imagined by Euclid, is called a *geometrical* point; such a dot as we must make in order to represent it, is called a *physical* point.

The word 'magnitude' is from the Latin *magnitudo*, size.

Examination I.

1. Give the derivation and your own explanation of the word 'definition.' 2. Say when a definition becomes redundant; also when it is deficient. 3. If you have given a correct definition of a square without mentioning more than one corner, what sort of definition will this become if the

statement in it about the one corner be changed into a similar one about all the corners? 4. When making a dot on a blackboard with chalk, you leave something on the board which can be felt with the finger. Will such a dot have parts? 5. Would you expect a dot of ink dried on the paper to have parts, even if you could not feel it with your finger? 6. A dot on paper is not truly a geometrical point; then of what use is it to make such a dot and call it a point? 7. Distinguish between a 'geometrical' or 'mathematical point' and a 'physical' one. 8. Give one or two other English words for 'magnitude,' and also the Latin. 9. If the word 'or' in Definition 1 be changed into 'and,' show that the definition would become redundant.

3. Lines.

2. A line is length without breadth.

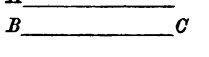
3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

A *physical* line *has* breadth, however little, and even some thickness or depth; we could not *see* its position otherwise.

The word 'even' is usually applied to surfaces; an 'even surface' meaning a perfectly flat one; that is, one in which there are no 'ups and downs.' So an 'even line' is one in which there are no 'ins and outs' to the right or left in going along it.

To express a straight line, we put letters to it; thus, A or BC. We usually require two letters placed at the two ends.



Examination II.

1. Explain the difference between a 'mathematical' and a 'physical' line. 2. Are the extremities the only points in a line? 3. What will the place of meeting or crossing of two mathematical lines form? 4. Imagine yourself starting at one end of a line to go *along it* to the other, and suppose that, in walking the whole distance, you never change your *direction*, what sort of line will you consider it to be? 5. Suppose, on the other hand, while always looking along the line just in front of you, you find yourself now and then looking away from the end you are going to, would you then consider the line to lie evenly between its extreme points? 6. If A be placed at one point, and B at another, what would the straight line joining them be called?

4. Surfaces.

5. A superficies, or surface, is that which has only length and breadth.

6. The extremities of a superficies are lines.

7. A plane superficies is that in which, any two points being taken, the straight line between lies wholly in that superficies.

The words 'superficies' and 'surface' are two words both derived from the same Latin words—*super*, above, and *facies*, a face. Their meaning is very much the same; but 'surface' is the word in common use for outward faces of things; and 'superficies' is chiefly used for *imaginary* surfaces in the sciences,—especially Geometry. A 'physical' superficies will have *some* thickness, however little.

The word 'plane' is from the Latin *planus*, even or level; and a plane superficies may be familiarly described as a perfectly flat surface, either really existing, or only imagined to exist, anywhere we choose in space. It seldom gets its full name, but is usually called a 'plane' simply.

Imagine we are watching a stone-mason at work upon the surface of a block of stone, trying to make it even. How does he test, when he wishes to do so, the evenness of the surface? He has a bar of wood with one of its edges forming a tolerably straight line. This straight edge he applies to the surface, and then looks at it from one side, trying to see between it and the surface. If he cannot see between them, the straight edge forms a straight line which lies, from one end of it to the other, '*wholly*' in the surface. But it is not sufficient to try the straight edge thus in one position only. Afterwards the workman places it in another position, and tries to make it join two *other* points in the surface. If he can do so, he tries another position, and another, and another, and so on, till he is satisfied. This workman's test is evidently in strict accordance with our definition.

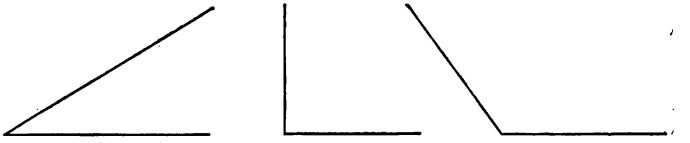
Examination III.

1. Give the derivation of the word 'plane': and compare its spelling with that of a geographical term of similar meaning. 2. Which of the words, 'superficies' and 'surface,' is most unlike the common origin of the two? Which is in more common use? 3. Give the usual abbreviated form of 'plane superficies.' 4. Describe a practical mode of testing evenness of surface, which illustrates Def. 7.

5. Angles.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



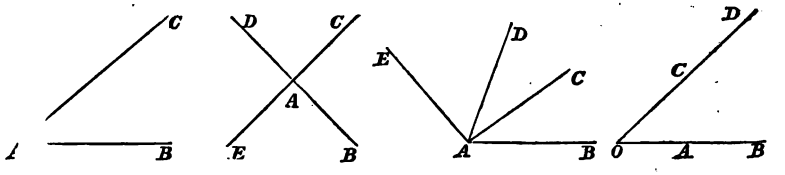
Notice three points in which this last definition differs from the one before it. First, the word 'rectilineal' is introduced, which means "contained by straight lines." Secondly, 'straight' is introduced, as is required by the previous term 'rectilineal.' Thirdly, the words "in a plane" are left out; for they are expressed, in accordance with Def. 8, by the adjective 'plane' which precedes 'rectilineal.'

The word 'angle' is from the Latin *angulus*, a corner; 'rectilineal' from *rectus*, right or straight, and *linea*, a line. In future, 'rectilineal' will nearly always be *understood* before 'angle.'

Definition 9 does not express the real nature of an angle as well as could be done; if, indeed, it can be said to express it at all. Instead of 'inclination *to*' one another, it would be better to say '*from*' one another; and it would be better still to say '*divergence* from.' To make a line *in*cline more to another, is to make the angle *less;* while to make two lines *di*verge more, is to make the angle between them more also. We shall find another argument to urge against the same definition on coming to Definition 10.

It is important to notice the different ways in which angles may be expressed when we come to discuss them.

When two lines meet without cutting, as *AB*, *AC*, in the first of the next figures, the angle between them may be denoted by the single letter placed at the corner, as *A*; or, also, by the three letters which are on the two lines, but keeping *A* in the middle, as *BAC* or *CAB*.



When two straight lines intersect or cut one another, as *BD*, *EC*, inter-

sect at A, we have four angles, each of which may be denoted by three letters, as just shown. Thus, the angle between AC and AD is CAD or DAC. We might also sometimes denote the angle by one letter placed *within* the corner, as BAE in the same figure by A.

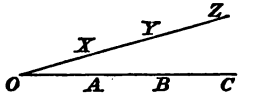
When more than two lines meet at a point, we have various angles, each denoted in a similar manner, as BAC, CAE, &c.

When either of the straight lines containing an angle has more letters than one along it, we may take any one of them for our purpose. Thus, the angle O, in the last figure above, may be denoted by AOC, AOD, BOC, or BOD.

This arises from a fact which should be carefully remembered, namely, that the length of either line containing an angle has nothing to do with the magnitude of the angle. Thus, OC above has the same inclination to OA as OD has, and so on. We may state the fact otherwise by saying that, *to lengthen one of two lines which meet, does not alter the divergence of the one from the other.*

Examination IV.

1. What does 'rectilineal' mean? Give its derivation, and that of 'angle.' 2. Give an instance of abbreviation of language in Def. 8. 3. State the three points in which Defs. 8 and 9 differ. 4. What word in Def. 9 will usually be left understood afterwards? 5. Explain the use we make of Def. 8 in stating Def. 9. 6. What little word might be changed in these definitions with advantage? 7. Give a definition of 'rectilineal angle,' using the words 'divergence' and 'from' instead of 'inclination' and 'to.' 8. Write out each of the angles formed by the straight lines BD and CE when they intersect in A, as in the second figure of Art. 5. 9. Write out six angles formed in the third figure of the same, each in two ways. 10. Write out the adjoining angle in as many different ways as you can, and say how many.

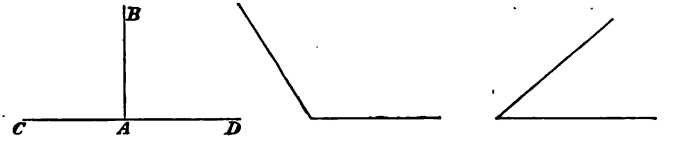


6. Kinds of Angles.

10. When a straight line standing on another straight line makes the adjacent angles equal to each other, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.



'Adjacent' means lying next to, from the Latin *ad*, to, and *jaceo*, I lie. Adjacent angles have one of their bounding lines in common. Thus, each of the two adjacent angles *BAC*, *BAD* has *BA* for one boundary.

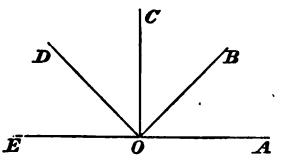
Now, referring to Def. 9 once more, when *BA* stands upon *CD* in the way described in Def. 10, it cannot properly be said to be *inclined* at all to the line *AD* or *AC*. In other words, *BAD* and *BAC* are not angles as defined in Def. 9. The meaning of that definition is plain enough, and yet it would be better if we could find no objection to the words. But we must not linger on the point any more, except to point out to the student that he may derive some warning from it how much care is often necessary in order to ensure that our language shall really express our meaning.

Def. 10 is a double one, as it defines two things, that is, 'right angle' and 'perpendicular.' We may define 'right angle' alone by stopping at the semicolon. But, to define 'perpendicular' apart, we must say something like this:—"When a straight line stands on another so as to make the adjacent angles equal, either of the lines is called a perpendicular to the other."

The word 'perpendicular' is from the Latin *perpendiculum*, a plumb-line.

EXAMINATION V.

1. Explain the word 'adjacent'; give its derivation, and also that of 'perpendicular.' 2. In the second figure of Art. **5** name an angle on each side of *BAC* adjacent to it. 3. Similarly for *CAD* in the same figure. 4. Write out the definition of a 'right angle.' 5. Define a 'perpendicular.' 6. If *BOD* be a right angle, say whether *AOD* is obtuse or acute; also *BOC*, *BOE*, and *COD* respectively.



7. FIGURES.

13. A term or boundary is the extremity of anything.

14. A figure is that which is enclosed by one or more boundaries.

Definition 13 is merely an ordinary explanation of one word by another; such as any dictionary would give.

A figure might be more clearly described as a *portion of space* closed in by boundaries. This space, in "Plane Geometry," is also assumed to be *plane* space, or space in some plane surface real or imaginary. And therefore our 'figures' will always be '*plane* figures,' or figures in a plane. The purpose of Plane Geometry is to prove by argument numerous important facts which have been discovered relating to *plane* figures, angles, and lines.

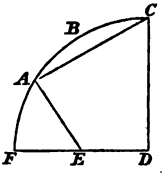
The boundaries of figures may be straight lines or curved lines, or some of one kind and some of the other. When there are more boundaries than one, each of them may be called a *side*.

If we wish to name a figure, we put letters along its boundary or boundaries, more especially at the corners, if there are any, and use three or more of these letters, taking care to use just enough to distinguish the figure from any other which may have some of the same letters as itself. Thus the circle in Diagram I., which has *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *K* along its circumference, may be called the circle *ABC* or *BCD* or *ADH*, or we may thus use any other three of the letters named. Sometimes we may have reason to use four or more of the same letters. It is the same with all other kinds of figures.

The word 'figure' is sometimes used for the *boundary* all round the figure, and then the space enclosed is called the *area* of the figure.

Examination VI.

1. How many figures are contained by the straight lines which form the capital letter *A*? 2. How many figures respectively in *B*, *D*, *O*, 8? 3. How many in the letters of the word 'dog'? 4. Name the separate boundaries of the figure *ABCDE* in the diagram. 5. Also of the figure *ABCDEF*. 6. And of the figure *ACDE*. 7. What is the word 'area' used for, and what does 'figure' then mean?

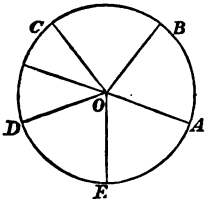


8. Circles.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.



The most important of all figures is defined first, although bounded by a curved line (and not by "one straight line," as some beginners will inadvertently say!).

This is another double definition, defining 'circle' and also its 'circumference.'

Definition 16 might have been similarly incorporated with Def. 15, and *by itself*, of course, is no definition of 'centre of a circle.'

It would be well to learn, along with Defs. 15—17, the following one:

16 (*a*) Each of the equal straight lines which may be drawn from the centre to the circumference of a circle is called a *radius*.

The word 'circle' is often used instead of the longer word 'circumference,' when there is no danger of one's being misunderstood in doing so.

For the derivations of some words now introduced, we have—Latin *circulus*, a ring; *circum*, around, and *fero*, I carry; *radius*, a spoke of a wheel; Greek *dia*, through, and *metron*, a measure.

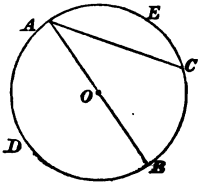
EXAMINATION VII.

1. Give the derivations of 'circle,' 'circumference,' 'diameter,' 'radius,' and say which of these words is often used in place of another. 2. Distinguish between the circle *ABCDE* and the circumference *ABCDE*. 3. Define 'circumference' without saying more than is necessary. 4. Define 'centre of a circle,' and do not say too little. 5. If a straight line be drawn from a point in the circumference to the centre, what may it be called? 6. If the same straight line run through the centre until it meet the circumference again, what will then be its name? 7. Name all the radii in the figure of Defs. 15—17. 8. How many radii are equal to one diameter?

9. PARTS OF CIRCLES.

18. A semicircle is the figure contained by a diameter, and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line, and the part of the circumference which it cuts off.



Definition 18 is not used until we are nearly at the end of Book II., and could be dispensed with even there. As *semi* means 'half,' in using the name 'semicircle' we are *assuming* that the diameter of a circle cuts it into halves. If we follow the common practice of Euclid, this should *not* have been assumed, for the simple reason that we are able to prove it; as the learner will perhaps notice on reading Euclid's Book III.

Definition 19 is not used until Book III., and is repeated among the definitions of that book.

A portion of a circumference is often called an 'arc.'

For reasons just stated, the above definitions would have been better reserved for the Third Book, in which the circle is more especially studied.

10. Rectilineal Figures.

20. Rectilineal figures are those which are contained by straight lines.

Kinds of Rectilineal Figures.

21. Trilateral figures, or triangles, by three straight lines.

22. Quadrilateral figures by four straight lines.

23. Multilateral figures, or polygons, by more than four straight lines.

'Trilateral,' from Latin *tres*, three, and *latus*, a side, means three-sided; 'triangle' means three-angled.

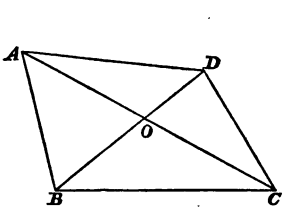
'Quadrilateral' means four-sided; 'multilateral' many-sided; 'polygon' many-angled.

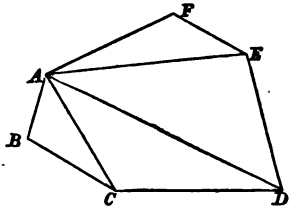
The Latin *quatuor* means four, *multi*, many or several; the Greek *polus*, many, and *gonia*, a corner.

Examination VIII.

1. What is the name given to the space between the straight line AC and the arc AEC in the figure of Art. 9? 2. Also to the space between

AOB and *ADB* in the same, *O* being centre? 3. Define an arc. 4. Connect the derivations and meanings in respect to 'trilateral,' 'quadrilateral,' 'multilateral,' 'polygon.' 5. State or write the name in letters of the triangle *ABC* in five other different ways. 6. Write down eight triangles which occur in the figure *ABCD*. 7. Write three quadrilaterals occurring in the figure *ABCDEF*. 8. Write two five-sided polygons in the same figure, and one six-sided polygon.





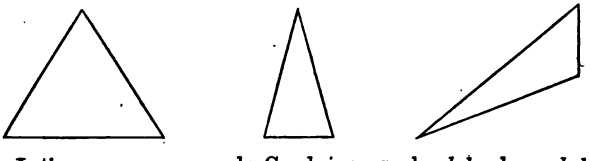
11. TRIANGLES NAMED ACCORDING TO THEIR SIDES.

Of three-sided figures—

24. An equilateral triangle is that which has three equal sides.

25. An isosceles triangle is that which has two sides equal.

26. A scalene triangle is that which has three unequal sides.



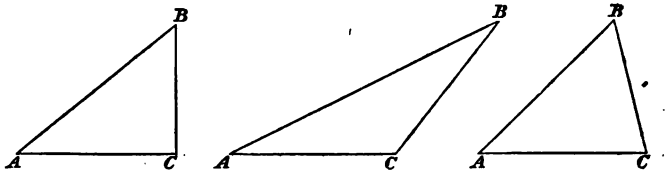
The Latin *æquus* means equal; Greek *isos*, equal; *skelos*, leg; *skalenos*, limping, uneven.

12. TRIANGLES NAMED ACCORDING TO THEIR ANGLES.

27. A right-angled triangle is that which has a right angle.

28. An obtuse-angled triangle is that which has an obtuse angle.

29. An acute-angled triangle is that which has three acute angles.



In Latin, *obtusus* means blunt; *acutus*, sharp.

More will be said about these definitions afterwards (note, Art. **65**). If it were not for the advantage of having all the definitions in one collection for easy reference, it would be better to take these definitions themselves and some others afterwards.

The *base* of a triangle, as the name would suggest, is usually the lowest side. Still, *whenever two sides of a triangle have been named, the remaining side is often called the base, whichever it may be.* In an isosceles triangle, we usually call those which are equal 'the sides.'

Notice the word 'three' in Def. 29.

The side opposite the right angle of a right-angled triangle is called the 'hypotenuse.'

Examination IX.

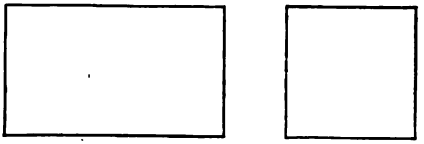
1. Familiarly explain 'equilateral,' 'isosceles,' 'scalene,' having regard to their derivations. 2. Is an equilateral triangle isosceles? 3. Given that AC is to be called the base of some triangle ABC, name the sides. 4. If AC, BC be called the sides of the triangle ABC, what is AB to be called? 5. In the triangle ABC, supposing that AB and AC are equal, name the sides and base.

13. Kinds of Quadrilaterals.

Of four-sided figures—

30. A square is that which has all its sides equal and all its angles right angles.

31. An oblong is that which has all its angles right angles but not all its sides equal.



32. A rhombus is that which has all its sides equal, but its angles are not right angles.

33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal nor its angles right angles.

34. All other four-sided figures besides these are called trapeziums.



With respect to Definition 30, if a figure has four equal sides and *one* angle a right angle, we can *prove* that all the other angles *must be* right angles; so that this definition says too much, and, for that reason, is not a good definition. It would be correct to state it thus,—"A square is a four-sided figure which has all its sides equal and one angle a right angle." The learner knows well enough what a square is like, and will probably think Euclid's description is better than this; but he should remember that a description and a definition are not necessarily the same thing at all.

Figures which have all their angles right are sometimes called 'rectangles.' This term generally takes the place of 'oblong.'

Definition 34 is not much observed in modern books. Euclid employs it once in Book I.

EXAMINATION X.

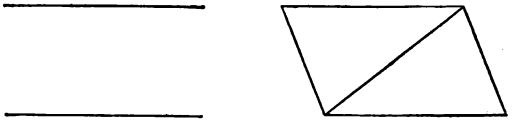
1. State in what respect Euclid's definition of a square is redundant. 2. Give two classes of figures included under the name 'rectangle.' 3. What name is more common than 'oblong' for that figure? 4. In what respect does an oblong differ from a square? 5. In what respect does a rhombus differ from a square? 6. How does a rhomboid differ from a rhombus?

14. PARALLELS AND PARALLELOGRAMS.

35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.

(A.) A parallelogram is a four-sided figure whose opposite sides are parallel.

(B.) A diagonal of a parallelogram is a straight line joining two opposite angles. It is sometimes also called a diameter, and the word is used similarly for other quadrilaterals.



Definitions *A* and *B* are added to those of Euclid, as the words they define are in common use in his Elements.

We shall be able to show afterwards that squares, oblongs, rhombuses, and rhomboids are different classes of parallelograms; but we are not to assume this to be so, until we have given the proofs.

A trapezium having two sides parallel is called a 'trapezoid.' The word trapezium itself is sometimes used with this meaning.

Obs.—The next three Examinations are intended to be taken in conjunction with a revision of the definitions as given in the larger type. They will refer chiefly to the diagrams at the end of the book.

Examination XI.

(The questions beginning at 5 are on Diagram I.)

1. What sort of point is it which possesses surface? 2. Plato said that a straight line is such that, if the eye be placed in a continuation of it, one extremity hides all the rest. Could this be employed as a test of a geometrical line? Give a reason. 3. Is there any kind of boundary which has no magnitude? 4. The definition we have given as 7 is due to Hero the Elder. Euclid himself says:—"A plane surface is that which lies evenly with the straight lines in it." State which other definition is uniform in manner with this. 5. Mention two angles adjacent to the angle AOB; also two adjacent to POQ and to ROU. 6. Name two angles which have OA as a common bounding line; and similarly for OR, OD, OH. 7. Write in four different ways the angle which is between OP, OB. 8. Write the angle between OP, OQ in four different ways, using different sets of letters each time. 9. Write the three angles which are separate parts of the angle POR. 10. Write two others, each of which is a part of the same angle POR. 11. If POR be a right angle, of what kind is POQ, and also POS? 12. Write down the terms of the figure $POVX$. 13. If POT and QOV are both diameters, name four different semicircles of the larger circle. 14. Count the number of small segments cut off along the edge of the same circle.

Examination XII.

(Diagram I. is referred to.)

1. If *HK*, *KX*, *XH* are all equal, what sort of triangle is *HKX*? 2. If *PQ* is equal to *OP*, what sort of triangle is *POQ*? 3. Give the names, in letters, of eight isosceles triangles in the circle *ABCDH*. 4. *TE* is not equal to the radius of either circle, nor is *TG*; to what sort of triangles do *OET* and *OGT* belong? 5. Distinguish between the angle *HOK* and triangle *HOK*. 6. Are the triangles *HOK* and *OKH* different? 7. Are the angles *HOK* and *OKH* different? 8. How many different triangles, and how many different angles, may be denoted by the following:—*COD*, *COR*, *ODC*, *ORC*, *DRC*, *RCD*, *RCO*? 9. Suppose *OEU* to be a right angle, then name the triangles *EOU*, *EOT* according to the nature of their angles. 10. Which side of the triangle *EOU* is the hypotenuse? 11. If, also, each of the angles *EFO*, *FOE* is acute, say why *FOE* must be an acute-angled triangle. 12. *OH* is hypotenuse of the triangle *OHL*, and *OK* of the triangle *OKL*; name a right angle in each of those triangles.

Examination XIII.

(For Questions 1—6 see Diagram II., and for 7—9 see Diagram III.)

1. *AD* and *PV* are parallel, and so are *PV* and *BC*; will *AD* meet *BC* if both are produced ever so far both ways? Say why. 2. Show that, if *PV* and *BC* are each parallel to *QU*, they cannot meet one another. 3. *AB*, *YR*, *XS*, *DC* are all parallel; also *AD*, *PV*, *QU*, *BC*; also *PY*, *QX*, *BD*, *RV*, *SU*. Say what kind of figure each of these is—*APEY*, *PQXY*, *PHDY*, *ABSX*, *BSUD*—remembering that all the sides of the little triangles are equal. 4. If *A* were a right angle, what would *APEY* become then? 5. Write the name of angle *A* in ten different ways, using a different set of letters each time. In how many more ways can you write the same angle? 6. Write down the geometrical terms of *PQFHXY*. 7. All the angles at *E* are equal to one another; show that they must each be a right angle. 8. If a straight line joining *AC* passed through *E*, *G*, how many right-angled triangles would it form?—how many hypotenuses? 9. Which figure may be described as 'lozenge-shaped'? 10. From derivations previously given, infer that of 'diagonal.'

THE POSTULATES.

15. Instruments for the Construction of the Figures.

We shall require to draw many figures in order to help our reasoning; and for this purpose we might use various instruments, some of which the learner will most likely have seen. Euclid, however, only requires us to know something about the use of two, namely—some instrument having a *straight edge*, with which straight lines may be ruled, and when necessary lengthened; and a *pair of compasses*, or some other means of making a circle. At this stage, Euclid also proceeds to state *exactly* what he requires that we shall be able to do with these instruments. His demands are called 'Postulates,' which is from a Latin word *postulo*, I demand.

Postulates.

Let it be granted—

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre at any distance from that centre.

In Postulate 2, 'produced' means *lengthened out*.

In Postulate 3, 'described' means *drawn*.

After reading the above 'postulates,' or 'demands,' the learner may, perhaps, think that they do not ask for much. This is certainly true enough; for, in fact, just as little is asked for as can be made to answer the purpose. It is Euclid's general rule, in regard to the use of an instrument, to apply *neither less nor more* than he expressly demands or postulates. This point is dwelt upon in the questions which will now follow.

Examination XIV.

1. What is a 'postulate'? 2. What instrument will suffice to perform what is required by Postulate 1? 3. What instrument will suffice to perform what is required by Postulate 2? 4. What instrument also for Postulate 3? 5. Some 'straight edges' have marks upon them denoting length. Is there any authority in the postulates for using these? 6. Distances may be compared and also transferred by means of compasses. Is this permitted by any postulate? 7. Give the meaning of 'described,' in Postulate 3. 8. In the same postulate, what other word should strictly

be used instead of 'circle'? (see Art. 8). 9. Can the real circle be properly spoken of as at a distance from its centre? 10. What lines have their lengths equal to the distance of the circumference from the centre? (see Art 8.) When the compasses are in use, what two points of the compasses are at a distance from each other equal to one of these lines?

Note.—If the student does not possess some knowledge of Practical Geometry, let him use both straight-edge and compasses in drawing all figures until he arrives at the end of Proposition III. Afterwards he may proceed, as is the usual way, to illustrate his work by freehand diagrams.

16. Some Applications of the Postulates.

Postulate 1.—Take two points at random; put A at one, and B at the other. By Postulate 1, we may *join A to B by a straight line.* This is usually expressed briefly by saying "join AB."

Postulate 3 is often expressed more briefly by using the word 'radius,' thus: "A circle may be described with any centre and any radius."

Examination XV.

1. What words are understood in the direction "join AB"? 2. Take two points about half an inch apart on your paper, and call one A, the other B. Join AB. What postulate have you now applied? 3. Lengthen out AB a little beyond B to some point, and call this other point F. What postulate have you now applied? 4. With centre A and radius AB describe a circle (or circumference), and place the letters C, D along it. What may this circle be called, and what postulate has been applied in drawing it? 5. With the same centre A and a radius AF which is greater than AB describe another circle. Join AC, and produce it until it meets the other circle in some point; which call G. What postulates have been applied in these operations? 6. In the same figure join AD; produce AD to meet the outer circle in a point H. Join BC, CD, FG, GH. How often has Postulate 1 been applied in these operations?

Note.—The student may verify his construction by comparing it with the figure which accompanies the next Exercise.

AXIOMS.

17. The Foundations of Reasoning in Geometry.

Numerous facts are known relating to geometrical figures; and Euclid's object is to establish the most important of these by sound

argument. In doing so, he uses some facts to lead us on to others; and from these he steps on to others again. In this process there must, of course, be a beginning—in other words, some facts must themselves be assumed as true; from which we may then proceed to deduce all the rest, which constitute the science of Geometry. Facts adopted in this way as true without proof are usually called *axioms*. It is clearly necessary that they should be chosen with special care and skill. Euclid performs this service for us in Geometry; and in selecting his axioms he appears to adopt as his rule that each one must be, firstly, *required;* secondly, *self-evident*, that is, obvious to ordinary minds without proof; and thirdly, *incapable of being proved.*

18. Common Notions.

The first seven axioms apply to all kinds of magnitude whether geometrical or not; and Euclid himself calls them "common notions," that is, notions which people in general have in their minds.

Axioms 1—7.

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are double of the same, are equal to one another.
7. Things which are halves of the same, are equal to one another.

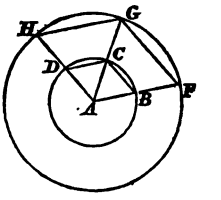
Although these are all so easy to understand, it will be to our future advantage to work a number of simple Exercises upon them.

When applying Axioms 4 and 5, we usually happen to know which is the *greater* of the pair of unequals; and for practical purposes these axioms might advantageously be stated as follows:—

4. If equals be added to unequals, the wholes are unequal, that being the greater which contains the greater of the first unequals.

5. If equals be taken from unequals, the remainders are unequal, that being the greater which contains the greater of the first unequals.

EXERCISE I.



1. If DH, in the diagram annexed, is equal to AB, prove that AC is also equal to DH.

The answer to this will be fully given as a *model* to assist in working some Exercises which follow it.

Proof.

It is given that DH is equal to AB;

and because A is the centre of the circle BCD, therefore AC is also equal to AB. (Def. 15)

But things which are equal to the same thing are equal to one another. (Ax. 1)

Therefore DH and AC are equal to one another.

Which was to be proved.

2. If CG is equal to AB, prove that AC is equal to CG.

(Let the proof be in little paragraphs like the one just given, and give the references in parentheses.)

3. If AC is equal to CD, prove that AD is equal to CD.

Say what kind of triangle ACD must then be.

4. If AH is equal to HG, show that the triangle AHG must be equilateral.

Nos. 5—9 refer to Diagram I.

5. Given that OA and AC are each equal to UV, prove that the triangle AOC must be equilateral.

The answer will be given here as another model.

Proof.

It is granted that OA and AC are each equal to UV.

Therefore AC is equal to OA. (Ax. 1)

And since O is the centre of the circle ACE,

therefore OC is also equal to OA. (Def. 15)

Therefore OC is equal to AC; (Ax. 1)

hence OA, OC, and AC are all equal to one another.

Therefore the triangle OAC is equilateral. (Def. 24)

6. Given that PQ is equal to OR; prove that the triangle OPQ is equilateral.

7. Given also EC is equal to OA; prove the triangle EOC equilateral.

8. Given that OPQ is an equilateral triangle; show that PQ must be equal to OS.

9. Given that CEO and CER are both equilateral; prove the quadrilateral $OCRE$ equilateral.

Nos. 10—17 refer to Diagram II., in which it must be remembered all the sides of the small triangles are equal.

10. Show that AQ is equal to AX.

Proof.

Since AP is equal to AY,

and PQ is equal to YX;

therefore the whole which is formed by adding PQ *to* AP *is equal to the whole formed by adding* YX *to* AY; (Ax. 2)

that is, AQ is equal to AX.

11. Given AB is equal to AD; prove that BP is equal to DY.

12. Given that YZ is less than XT; show that HT is greater than EZ.

13. Given FL less than GM; prove RL less than SM.

14. Prove AQ equal to BP.

15. Given LF half of RE, and EZ half of FY; prove LF and EZ equal.

16. Given that quadrilateral $APEY$ is equal to twice triangle PEY, and $PQEY$ is double of the same triangle; prove that $APEY$ is equal to $PQEY$.

17. If FE bisects the quadrilateral $QFHE$, and FH bisects $EFGH$; prove the triangles FQE and FGH are equal.

19. Modified Use of some Axioms.

We have already referred to modifications of Axioms 4 and 5, in which account is taken of the lesser and the greater in each pair of unequals. We may, and often do, apply other modifications of the axioms, such as the following; which are respectively related to axioms 2, 3, 4, 5, 6, 7:—

If the same be added to equals, or if equals be added to the same, the wholes are equal.

If the same be taken from equals, or equals from the same, the remainders are equal.

If the same be added to unequals, the wholes will be unequal, that one being the greater which contains the greater of the former unequals.

If the same be taken from unequals, the remainders will be unequal, &c.

Things which are doubles of equals are equal.

Things which are halves of equals are equal.

The learner is not advised to commit these to memory, but simply read them attentively, before working the next exercise.

The word 'bisect' means to divide into two *equal* parts, that is, into halves.

One thing is said to be 'common' to two others when it belongs to both of them.

Exercise II.

(Diagram II. is referred to throughout.)

1. Given the triangles *APY*, *PEQ* equal, show the quadrilaterals *APEY*, *PQEY* are equal.

Proof.

It is given that the triangle *APY* is equal to the triangle *PEQ*; add to each the same triangle *PEY*;

then the whole *APEY* is equal to the whole *PQEY*. (Ax. 2)

2. Given the quadrilaterals *PQEY*, *PEXY* equal, prove the trapeziums *AQEY*, *APEX* are equal.

3. Given the quadrilaterals *EFGH*, *FGVH* equal, prove the triangles *EFH*, *GVH* equal.

4. Given the trapeziums *CUGR*, *CVGS* equal, prove the triangles *GRS*, *GUV* equal.

5. If *GS* bisects the quadrilateral *GRSU*, and *SU* bisects *CUGS*, prove the two quadrilaterals are equal.

6. The same being given, prove the triangles *SCU*, *GRS* equal.

7. If all the angles round the point *E* are equal to one another, prove the angles *PEX*, *FEX* are equal.

8. Also the angles *PEX*, *HEY*.

9. Given the angles *FGV*, *HGU* are equal, prove the angles *FGH*, *VGU* are also equal.

10. If all the angles of the triangle *FEH* are equal, also those of the triangle *FGH*, and also the angles *FEH*, *FGH*; prove the angles *EFG*, *EHG* are equal; and each double of the angle *FEH*.

20. Superposition.

Superposition means, literally, *the placing of one magnitude upon another;* which we may do when we wish to compare their sizes or magnitudes. This is a very familiar process, and Euclid adopts it as his test of equality in our next axiom.

This axiom is very important. The three succeeding ones are somewhat related to it, and are given with it.

Axioms 8—11.

8. Magnitudes which coincide with one another, that is, which exactly occupy the same space or position, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

The 8th axiom is often expressed thus:—"Magnitudes which coincide with one another, that is, *which exactly fill the same space*, are equal to one another." But the explanatory clause in italics cannot apply to angles and straight lines, which do not *fill* space.

Unless Axiom 10 were assumed, there would be a difficulty about applying Axiom 8 to rectilineal figures. For, to begin with a straight line, when we have fitted the two ends of one straight line upon the two ends of another, we could not be sure that the lines are *entirely* fitted one upon the other, if it were possible for them to enclose a space.

We shall find excellent practice in the applications of Axioms 8—10 in the Exercise preliminary to Proposition IV.

21. Parallelism.

In order to complete the collection of axioms, the 12th will now be given. It will be much better, however, not to discuss it until we have made some further progress in the subject.

12. If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Examination XVI. (Recapitulatory.)

1. What sort of facts are called axioms? 2. If a fact be very easy to believe without proof, but at the same time is capable of proof, would Euclid admit it as an axiom? 3. The first axiom speaks of 'things'; does this necessarily refer to things occurring in Geometry only? 4. Explain the words 'bisect'; and say what is meant when a magnitude is said to be common to two others. 5. State the meaning of 'superposition.' 6. To what use does Euclid put the process of superposition? 7. State another form of Axiom 8 which is often used, and show its defect. 8. What axiom may be used to show that the lesser of two given straight lines may be made to lie along a part of the greater?

THEOREMS AND PROBLEMS.

22. What are Propositions and Problems?

Euclid's reasoning, in his "Elements," is divided into numerous portions, generally called 'Propositions.' Strictly speaking, a *Proposition* is that which *places before us* the object of some piece of reasoning upon which we are about to enter. The word comes from the Latin *pro*, before, and *pono*, I place. But, just as a circle is defined to be one thing and often used for another, so the meaning of Proposition in Geometry is very commonly extended, so as to embrace the whole argument to which it belongs.

This extension of meaning brings into use another word, namely '*Enunciation*.' This is from the Latin *enuntio*, I announce or declare, and, in general, simply means a statement or declaration; while, in Euclid, it may be defined as follows:—

The Enunciation of a Proposition is the general statement of its object, or the purpose with which it commences.

There are two classes of Propositions, called *Problems* and *Theorems*.

We shall meet first with Problems.

A Problem is a Proposition which proposes that some geometrical construction shall be effected.

The Enunciation of a Problem is divided into two parts, namely, the *datum* (or, plural, *data*), and the *quaesitum* (or, plural, *quaesita*). In Latin, *datum* means something given, and *quaesitum* something sought. Hence

The data of a problem are the things which are given.

The quaesita of a problem are the things which are to be done or made.

The three postulates (Art. **15**) are sometimes spoken of as 'self-evident' problems.

'Solving' a problem means, finding out how to do what its enunciation requires us to do. In theoretical Geometry (such as Euclid's), after stating the solution of a problem, we usually add a *proof* that the solution is a correct one.

It is to be remembered, in solving our problems, that we are restricted to a *straight-edge* and *compasses* to work with.

A 'finite' straight line is limited in length, having two extremities given or understood.

Exercise III.

Distinguish in each of the following little problems 1—6 what is given from what is required; then give a solution and proof.

A full answer to No. 1 is given, which should be followed as a model in working Nos. 2—6, all definitions and postulates being carefully quoted in the margin, as shown.

1. To produce a finite straight line until it becomes twice as long as at first.

An answer to this might be given as follows:—

Given: a finite straight line.

Required: to double its length.

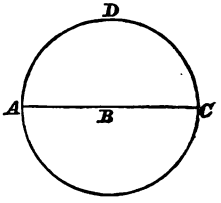
Solution.

Let AB be the given finite straight line.

Taking B as centre, and BA as radius, draw the circle ADC; (Post. 3)

and produce AB until it meets the circle again in the point C.

We have now done what was required.



Proof.

For, because B is the centre of the circle ADC, BC is equal to BA. (Def. 15)

Therefore AC is double of AB,

and may therefore be taken as the line required.

2. Given a circle, its centre, and a point in its circumference, to draw through this point a diameter.

3. Given two points, draw a circle which shall have one point as centre and the other in its circumference.

4. Given a circle and its centre, make an isosceles triangle having its vertex at the centre.

5. Given a finite straight line, to make an isosceles triangle with this line for one of its equal sides. (Use Post. 3 first.)

6. Given a finite straight line and its middle point, to draw a semicircle upon it.

7. Take two points A, B and join them; (Post. 1)

with centre A and radius AB describe a circle; (Post. 3)

and with centre B and radius BA describe another circle;

produce AB to meet one circumference in C and BA to meet the other in D. (Post. 2)

In the figure thus constructed, how many times is AB contained in CD?

By what authority may you assume BC equal to BA, and also AD equal to AB?

8. By what axiom may we infer that BC and AD are therefore equal?

SECTION I. PROPOSITIONS I.—VI.

First case of Equality of Two Triangles, with accessory Problems.

23. Proposition I.—Problem.

To describe an equilateral triangle upon a given finite straight line.

Let AB be the given finite straight line.

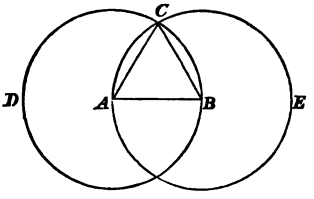
It is required to describe an equilateral triangle upon AB.

From the centre A, at the distance AB, describe the circle BCD; (Post. 3)

and from centre B, at the distance BA, describe the circle ACE, cutting the former circle in C;

and join CA, CB.

Then ABC shall be an equilateral triangle.



Because A is the centre of the circle BCD,

therefore AC is equal to AB; (Def. 15)

and because B is the centre of the circle ACE,

therefore BC is equal to BA;

wherefore AC and BC are each equal to AB.

But things which are equal to the same thing are equal to one another; (Ax. 1)

therefore AC is equal to BC;

hence AB, BC, CA are equal to one another.

Therefore the triangle ABC is equilateral; (Def. 24)

and it is described upon the given finite straight line AB.

Which was to be done.

Note.—In the Exercises, where figures are not given although referred to, the student is required to draw them from the given directions.

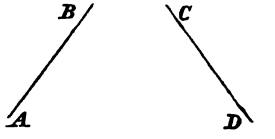
EXERCISE IV.

1. The circles of the foregoing figure intersect in another point besides C. Put F at this other point, and then prove, after the manner of Proposition I., that ABF is an equilateral triangle.

2. Prove also that $ACBF$ will be a rhombus.

3. Take a straight line inclined like AB, and construct an equilateral triangle upon it, on either side of it.

4. Also two equilateral triangles upon a straight line inclined like CD, one on each side of it.



5. In the diagram of Proposition I. produce AB both ways to meet the circumference in D and E. With centre A and radius AE describe a circle; also with centre B and radius BD describe a circle: let these circles intersect on H and join H to A and B. Then show that HAB is an isosceles triangle, stating the magnitude of each side as compared with its base.

24. THE CONSTRUCTION OF A PROPOSITION.

The *Construction* of a proposition describes the steps by which we complete the diagram so as to make it sufficient for the purposes of our reasoning.

It usually precedes the proof, but may be sometimes found partially or wholly interspersed among the steps of the reasoning.

When we wish to refer to the construction as authority for some subsequent statement, we place the abbreviation "Constr." in the margin.

Obs.—The letters Q. E. F. are the initials of the Latin "*Quod erat faciendum*," and mean "which was to be done."

25. PROPOSITION II.—PROBLEM.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given line.

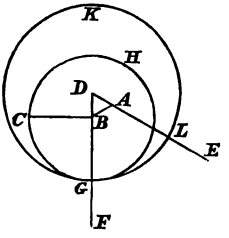
It is required to draw from A a straight line equal to BC.

Join AB; (Post. 1)

upon AB describe the equilateral triangle DAB, (I. 1)

and produce DA, DB to E and F; (Post. 2)
from the centre B, at the distance BC, describe the circle CGH; (Post. 3)
from centre D, at the distance DG, describe the circle GKL.

Then the straight line AL shall be equal to BC.



Because B is the centre of the circle CGH,
therefore BC is equal to BG; (Def. 15)
and because D is the centre of the circle GKL,
therefore DL is equal to DG;
and DA, DB, parts of these, are equal; (Constr.)
therefore the remainder AL is equal to the remainder BG. (Ax. 3)

But it was shown that BC is equal to BG;
therefore AL and BC are each equal to BG;
therefore AL is equal to BC. (Ax. 1)

Wherefore, from the given point A, a straight line AL has been drawn equal to the given straight line BC.

Q. E. F.

EXERCISE V.

1. Construct the diagram for Proposition II., similarly to the one which has been given, but making the equilateral triangle DAB on the other side of AB.

2. Perform the construction, as directed in the proposition, when A occurs in the continuation of CB.

3. Perform the same construction, placing A in BC itself.

[Other changes in the position of A may be made to furnish the learner with instructive exercises.]

4. Let BC be a given straight line; show how to draw from B a straight line equal to BC.

5. In the figure of Proposition I., produce AB both ways to meet the circles in D and E. Then prove that a straight line has thus been constructed which is equal to the sum of the three sides of the triangle.

6. Given any triangle, produce its base both ways to form a straight line equal to the sum of its sides.

Note.—The reasoning of Prop. III. will be found to be so easy that we need not give any preliminary consideration to it. Its importance must not, on that account, be under-estimated, so far as Euclid's Geometry is concerned. It will be found to meet with frequent application.

26. Proposition III.—Problem.

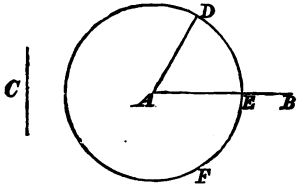
From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB a part equal to C.

From A draw the straight line AD equal to C; (I. 2)
and from the centre A, at the distance AD, describe the circle DEF. (Post. 3)

Then AE shall be equal to C.



Because A is the centre of the circle DEF,
therefore AE is equal to AD; (Def. 15)
but the straight line C is equal to AD; (Constr.)
therefore AE and C are each equal to AD;
wherefore the straight line AE is equal to C. (Ax. 1)

Therefore from AB, the greater of two given straight lines, a part AE has been cut off equal to the less.

Q. E. F.

EXAMINATION XVII.

1. Give the derivation of the word 'proposition,' and infer its *literal* meaning. What is its usual meaning in Geometry? 2. Give the derivation and use of Enunciation. 3. Name the two classes of propositions in Geometry, and define 'problem.' 4. What is meant by 'solving' a problem? 5. When is a straight line said to be finite? 6. What is the 'construction' of a proposition? 7. Explain the letters Q. E. F. 8. In the enunciation of Proposition I. separate what is *given* from what is *required*. 9. Similarly for Prop. II. 10. Also for Prop. III.

27. WHAT ARE THEOREMS?

Our Propositions now change character, and become 'Theorems.'

A Theorem is a Proposition which places before us some truth which it is proposed to demonstrate.

In the enunciation of a Theorem, there are always two parts, called the '*Hypothesis*,' and the '*Thesis*,' or '*Conclusion*.'

The Hypothesis of a theorem states one or more conditions which are to be granted.

The Thesis, or Conclusion, consists of an assertion which we propose to prove if the hypothesis be granted.

Thesis is a Greek word, meaning something *placed* or *set before us;* while the prefix *hypo* means *under*. We may, therefore, compare the hypothesis to a foundation upon which is built up the thesis.

In the course of the argument, it will be necessary from time to time to refer to the hypothesis, and this will be indicated by an abbreviation in parentheses, thus—(Hyp.).

As an example of separating hypothesis from thesis, let us take a simple theorem, namely—

If two radii of a circle form two sides of a triangle, the triangle will be isosceles.

Hypothesis: Two of the sides of a given triangle are radii of the same circle.

Thesis or Conclusion: That triangle is isosceles.

In the following exercise, the phrase 'each to each' means the same as 'respectively.' Thus, if A, B, C are equal to X, Y, Z, each to each; this means that A is equal to X, B equal to Y, and C equal to Z; or that the following are pairs of equal angles: A and X, B and Y, C and Z.

EXERCISE VI.

1. If two triangles ABC, DEF have two angles A and B of the one equal to the two D, E of the other, *each to each;* give the same equal angles *in pairs* of equals.

2. If angles A, B, C of the one triangle be equal to angles E, F, D of the other, each to each; write the equal angles in pairs.

3. If two sides AB, BC of the triangle ABC be equal to the two sides DE, EF of the triangle DEF, each to each; write those sides in pairs of equals.

4. Two sides of a triangle have been mentioned; what name may be then given to the third side?

5. Make ABC, DEF any two triangles, then write with three letters the angles opposite to AB, DE, AC, DF, BC, EF, respectively.

6. Distinguish between the angle BAC and the triangle BAC.

7. If we fit or apply one surface upon another, in order to try whether the two surfaces are equal, what axiom do we employ?

8. In the process of applying one straight line to another in order to see whether they are equal or not, distinguish three separate steps.

9. Referring to the figure of Proposition IV., suppose we have taken up the triangle ABC from its place and have put it down so as to have AB upon DE exactly, and we then find that AC will also just fall upon DF; what must be true respecting the angles A and D?

10. On the other hand, suppose we have AB upon DE, and we know that angle A is equal to angle D, where must AC fall if the triangle ABC be pressed down?

11. Suppose the boundaries of one figure fit upon the boundaries of another, what is the inference respecting their areas or surfaces inclosed?

12. Suppose we observe that, of two triangles ABC, DEF, we can make, at the same time, AB fall upon DE, AC fall upon DF, and BC fall upon EF; what may we say of the areas of these triangles?

13. Having thus found that AB and BC fit respectively upon DE, EF, draw the inference respecting the angles ABC, DEF.

Which sides must fit upon one another in order that we may infer that angles ACB and DFE are equal?

14. When B falls upon E, and C upon F, quote the axiom which shows the straight line BC must coincide with the straight line EF.

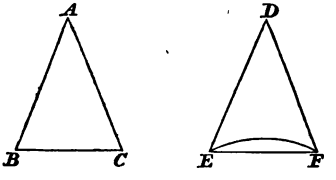
Note.—Although Proposition IV. may appear at first somewhat difficult, it is proved by very simple means. It is not that the proof is really difficult, but rather that its phraseology is apt to be only partially understood. On this account, if the student has not seen his way clearly in each step of the preceding exercise, it will be quite worth his while to work it through again, in whole or in part. He will then have practised every process which occurs in proving Proposition IV., and there only remains to build up the separate parts of the argument into one whole.

28. PROPOSITION IV.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal; then they shall have their bases equal, and the two triangles themselves shall be equal, and of their other angles those shall be equal to which equal sides are opposite.

Let ABC, DEF be two triangles which have the two sides AB, AC, and the angle BAC which they contain, respectively equal to the two sides DE, DF, and the angle EDF which these contain;
namely, AB equal to DE, AC to DF,
and the angle BAC to the angle EDF.

Then shall be equal also: the bases BC, EF;
the two triangles ABC, DEF;
the angles ABC, DEF, to which AC, DF are opposite: and
the angles ACB, DFE, to which AB, DE are opposite.



For, if the triangle ABC be applied to the triangle DEF, placing the point A upon D, and the straight line AB along DE;
then, because AB is equal to DE, (Hyp.)
the point B will coincide with the point E.
And because AB thus coincides with DE,
and the angle BAC is equal to the angle EDF, (Hyp.)
therefore AC will fall along DF;
also, because AC is equal to DF, (Hyp.)
therefore the point C will coincide with the point F.

Then, since B coincides with E, and C with F,
the base BC will fall along EF; or, otherwise, BC and EF will enclose a space; which is impossible. (Ax. 10)

Hence BC and EF coincide, therefore they are equal; (Ax. 8) *and the triangles ABC, DEF will then coincide;* therefore they are equal.

Also the angles ABC, DEF will coincide, and so must be equal; *and the angles ACB, DFE*, and therefore these are equal.

Wherefore, if two triangles have two sides, &c.,

Which was to be demonstrated.

29. On Applications of Proposition IV.

This proposition is put to very frequent use; and it is required to become perfectly familiar with two sets of things belonging to its enunciation. These are, the number and nature of its *conditions*, and the number and nature of the *consequences* which are said to follow of necessity from those conditions. As we have explained already, these two sections of the enunciation are called its hypothesis and its thesis. Learners very often fail to become clear about either hypothesis or thesis; so we may specially notice that the *hypothesis* of Proposition IV. is *three*fold, and its *thesis* or *conclusion* is *four*fold.

The hypothesis includes *three* equalities; namely,—

one side in one triangle equal to *one side* in the other,

a *second* side in one equal to a *second* side in the other,

and the *contained angle* of one triangle equal to the *contained* angle of the other.

Unless *all* these three conditions are granted, Proposition IV. proves nothing; and this fact is one which learners are rather apt to forget. The *third* is the condition most frequently overlooked; and, even when 'angle' is remembered, the word 'contained' is sometimes lost sight of.

The learner may easily make out for himself the four equalities contained in the conclusion.

Triangles which are equal in *seven* particulars, as *ABC, DEF* are shown to be in Prop. IV., are said to be equal *in all respects*. These seven particulars may be called 'elements' of the triangles; while the *six* angles and sides of a triangle are called its 'parts.'

30. Triangles with Parts in Common.

As before explained, that which belongs to two things at the same time is said to be *common* to them.

In the figure to Proposition IV. the triangles are perfectly distinct one from the other; and they have therefore nothing in common. Now examine the figure given in Exercise VII. to No. 3, in which are two triangles *ABC, ABD*. Name the sides of each triangle; and you will observe that

one side belongs to *both* triangles. This side is therefore said to be common to both triangles.

In the course of future investigations, we shall meet with triangles having parts in common very frequently; in fact, more frequently than not. They will be found, for example, in all the eight propositions which come next, except one; and it will therefore be wise to take some preliminary practice in treating them.

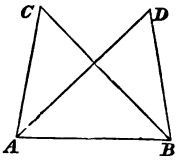
The letters *Q.E.D.*, which will be found at the end of Prop. V., are the initials of "*quod erat demonstrandum*"; and mean "which was to be demonstrated."

Exercise VII.

1. With respect to two triangles *ABC*, *XYZ*, let the hypothesis be that the sides *AB*, *AC* and angle *BAC* are equal to *XY*, *XZ*, and angle *YXZ*, each to each; write the fourfold conclusion.

2. Suppose in the triangles *ABC*, *DEF* the sides *AB*, *DF* are equal, also *AC*, *DE*, and angles *BAC*, *EDF*; state the conclusion so far as regards the other angles; that is, the third and fourth conclusions.

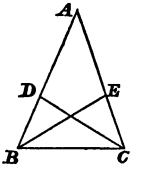
3. In the annexed figure, given that *CA*, *DB* are equal, name two sides of the triangle *CAB* which are respectively equal to two of the triangle *DAB*.



4. If the two angles *CAB*, *DBA* be also equal, state the four conclusions which may be drawn by the application of Prop. IV.

5. In the same figure, given *CA*, *CB* and the angle *C* equal to *DB*, *DA* and the angle *D*, each to each; deduce a threefold conclusion, and say why not fourfold.

6. In the annexed figure, given *ABC* is an isosceles triangle, and *BD* is made equal to *CE*; show *AD*, *AE* are equal.

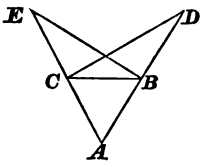


7. Find three parts of the triangle *ABE* equal, each to each, to three of the triangle *ACD*.

8. Draw four inferences by aid of Prop. IV.

9. Suppose it granted in addition that the angles *BDC* and *BEC* are equal, prove by Prop. IV. that the angles at the base of the triangle *ABC* are equal.

10. In the annexed figure, state the parts common to the two triangles *BCA*, *DCA*.



11. What is common to the triangles *DAC* and *EAB*?

12. Given, in these triangles, *AD*, *AE* equal, and also *AB*, *AC* equal; find three parts of the one respectively equal to three parts of the other.

13. Deduce a threefold conclusion, omitting that which relates to areas.

14. Given, in the same figure, AB, AC equal, and also AD, AE; prove BD, CE equal.

15. Given *also* BE, CD equal, and angles D, E; find three parts of the triangle DBC equal to three of the triangle ECB, each to each, without mentioning BC.

16. Deduce the conclusions relating to angles.

17. Take away the angle CBE from the angle ABE, and BCD from ACD; and state the remainders.

18. If ABE, ACD be equal, and also CBE, BCD, what will follow with regard to the two remainders?

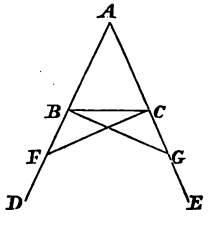
31. Proposition V.—Theorem.

The angles at the base of an isosceles triangle are equal to one another; and, if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles triangle whose sides AB, AC are equal; and let AB, AC be produced to D and E. (Post. 2)

Then the angles ABC, ACB shall be equal,
and also the angles DBC, ECB.

In BD take any point F;
from AE the greater cut off AG equal to AF the less; (I. 3)
and join FC, GB.



Because AB is equal to AC (Hyp.), and AG to AF; (Constr.)
therefore, in the triangles ABG, ACF,
the two sides BA, AG are equal to the two CA, AF, each to each,
and both pairs of sides contain the same angle FAG;
therefore the bases BG and FC are equal, (I. 4)

also the angles AGB, AFC respectively opposite to AB, AC,
and the angles ABG, ACF opposite to AG, AF.

And, because the whole AF is equal to the whole AG, (Constr.)
of which the parts AB, AC are equal,
therefore the remainders BF and CG are equal. (Ax. 3)

Hence, in the triangles BFC, CGB,
BF is equal to CG, and it was shown that FC is equal to GB,
and also that the included angles BFC, CGB are equal:
therefore the angles BCF, CBG opposite to BF and CG are equal,
and also the angles CBF, BCG opposite to FC, GB. (I. 4)

And, since it has been demonstrated
that the whole angles ABG and ACF are equal,
and that parts of them, the angles CBG, BCF, are equal;
therefore the remainders are equal, namely, the angles ABC, ACB, (Ax. 3)
which are the angles at the base of the triangle ABC.
And CBF, BCG have been shown equal,
which are the angles on the other side of the base.

Therefore, the angles at the base, &c. Q. E. D.

Corollary.—Hence every equilateral triangle is also equiangular.

For the meaning of this term 'corollary,' see the remarks which follow.

Note.—To remember the order of proof in Prop. V., notice that it discusses a pair of large triangles and a pair of smaller ones; and applies Axiom 3 after each discussion.

32. On Corollaries.

In the course of an extended piece of reasoning, such as that of Proposition V. or IV., we sometimes prove more truths than the one we have specially in view. More often still, other facts worthy of notice follow so easily from some part of the reasoning, or from the conclusion, that we perceive the truth of them at a glance, as soon as they are stated. For such inferences as these we have a name.

A Corollary to a proposition is an inference from its reasoning, or from its conclusion, which is so easy as to require little or no proof.

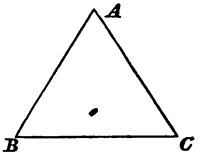
The abbreviation '*Cor.*' stands for corollary.

As an example to the student, we will prove the corollary which occurs above to Prop. V., leaving as practice for himself such as will occur hereafter.

Theorem : Every equilateral triangle is also equiangular.

Let ABC be an equilateral triangle.

All its angles A, B, C shall be equal.



Because AB is equal to AC; (Def. 24)
therefore the angle B is equal to the angle C. (I. 5)
Also, because BA is equal to BC;
therefore the angle A is equal to the angle C.
And because CA is equal to AB;
therefore the angle A is equal to the angle B.

Wherefore the angles A, B, C are all equal. Q. E. D.

33. Converse Theorems.

When we compare two theorems in respect to their hypothesis and thesis, we sometimes find that the hypothesis of one is the thesis of the other; and *vice versâ*. Such theorems are termed *Converse* Theorems. As examples, compare the two theorems :—

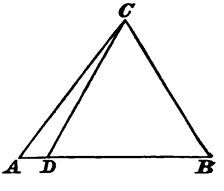
1. If two triangles be equilateral, they are also equiangular.
2. If two triangles be equiangular, they are also equilateral.

It will be desirable to say something more about converse theorems on another occasion.

That side of a triangle which is opposite to any angle, is said to '*subtend*' it. Literally, the word '*subtend*' means '*to stretch under*,' or '*in front of*.'

Exercise VIII.

(The same figure is referred to throughout.)



1. In the figure annexed, given CA equal to CB, name two equal angles.
2. If DC, DB are equal, name another pair of equal angles.
3. If DC is equal to DB, prove that AC cannot be equal to AB.
4. If it be certain that, if something called A is like B, then X will be like Y; but, on examining X and Y, we see they are not alike; will A and B be alike or not?
5. If it be certain that, if two magnitudes X and Y are equal, then the triangle CDB in the figure will be as large as the triangle CAB; what may we then say about X and Y?
6. In the figure, looking at ACB and DBC as separate triangles, what side have they in common?
7. It is also given, that DB is equal to AC; now write two sides of the one triangle equal to two of the other.
8. Will this be sufficient to draw any conclusion from by the aid of Op. IV.? If not, say what more will be required.

9. Suppose now it is asserted that *DB*, *BC*, and the angle *DBC* of the triangle *DCB* are respectively equal to *AC*, *CB*, and the angle *ACB* of the triangle *ACB*; then, by Prop. IV., draw an inference respecting the *areas* of the triangles *DBC*, *ACB*. Quote an axiom by which such a conclusion in this case would be contradicted. What is the conclusion you come to on the point?

10. State the sides which respectively subtend the angles *BCD*, *BCA*, *BDC*, *ADC*. What angles are both subtended by *CD*?

34. Proposition VI.—Theorem.

If two angles of a triangle be equal, the sides which subtend them shall also be equal.

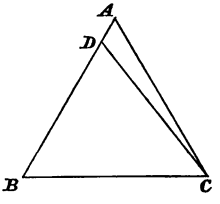
Let *ABC* be a triangle, having the angles *ABC*, *ACB* equal. *Then the sides AB, AC opposite them shall be equal also.*

For, if *AB* be not equal to *AC*, one of them must be greater than the other.

If possible, let *AB* be the greater.

From *BA* cut off *BD* equal to *CA*, (I. 3)

and join *DC*. (Post. 1)



Then, in the triangles *DBC*, *ACB*,

DB is equal to *AC*, (Constr.)

and *BC* common to both triangles;

therefore the two sides *DB*, *BC* are equal to the two sides *AC*, *CB*, each to each;

and the angle *DBC* is equal to the angle *ACB*. (Hyp.)

Therefore the triangle DBC is equal to the triangle ACB, (I. 4) the less to the greater; which is absurd.

Hence, *AB* and *AC* are *not* unequal;

that is, they are equal.

Wherefore, if two angles, &c. Q. E. D.

Cor.—Every equiangular triangle is also equilateral.

Exercise IX.

1. Given, in the triangle ABC, that CBA, CAB are equal angles, name the sides which are equal by Prop. VI.

2. Given, that BCA, BAC are also equal angles, name another pair of equal sides.

3. In the triangle XYZ, angles X, Z are equal; which sides are equal?

4. Prove the corollary to Prop. VI.

5. Let BG and CF in the figure of Prop. V. meet in H, then show that HBC is an isosceles triangle. (Assume what is proved in Prop. V. about the triangles BCF and CBG.)

35. Indirect Proof.

In the early portions of Geometry, we make our steps so simple that it is easy to see that the way in which they are deduced one from another is right. Certainly, it would be an advantage, in regard to this point, to know something of Logic. But we may feel safe enough for the present, if we are careful, and keep Euclid's propositions well before us as examples.

Assuming that we keep our mode of reasoning correct, if we commence our argument with a statement of some *fact or facts*, then every one of its steps will also form a *true* statement. Hence, if we should come to a step in the reasoning which we know is *not* true, we infer at once that the statement we started with will be also untrue. In other words, the hypothesis of the argument is wrong.

It may happen that we wished to prove this particular hypothesis wrong, in order to prove that some other statement is right. If so, our proof of the latter is called an *indirect* proof.

Suppose we say: "Here is a piece of glistening white substance; it is sugar." In saying this, we have adopted a thesis, namely, *the substance is sugar*. Suppose we proceed to test this thesis, by making it the basis or *hypothesis* of further reasoning, and say,—"We will put it into hot tea, and the tea will then taste sweet." Afterwards, however, we make a trial of the tea, and find it not sweet at all, but having a briny flavour. What must be our conclusion? It must be that our present hypothesis is false; for we know now the substance is not sugar. Instead of calling it false, however, we might call it a *trial* hypothesis, or a *trial* thesis in regard to the original proposition.

This kind of reasoning is often called *experimental*. In Geometry we argue in a way which is very similar to it, when we call the reasoning 'indirect.'

The proof of Proposition VI. is indirect; and the learner may now, perhaps, make out its false thesis, the reasoning based upon it as a new

hypothesis, the false conclusion we are led to, and the final or true conclusion.

Def.—An indirect proof of a proposition is one which shows that any hypothesis contradicting it could be made to lead to some false conclusion.

The student may hereafter observe that the second of two *converse* theorems is usually proved indirectly.

EXERCISE X.—RECAPITULATORY.

(In 1—13, Diagram I. is referred to.)

1. If the angles *COE*, *HOK* are equal, find three 'parts' of the triangle *COE* equal to three of *HOK*, each to each, so as to correspond with the hypothesis of Prop. IV.

2. Draw the conclusions established for these triangles by the same proposition.

3. Given that *OL* bisects the angle *HOK*, find three elements of the triangle *HOL* and three of the triangle *KOL* which are equal in pairs.

4. Deduce four other equalities in respect to the same triangles.

5. State the common part of the two lines *AT*, *FP*.

6. State what angle is common to the two angles *AOC*, *BOD*?

7. Also state the side which is common to the two triangles *HOG*, *HOK*.

8. Also the angle which is common to the two triangles *EOF* and *EOT*.

9. And the angle common to the three triangles *CDO*, *CRO*, and *QOR*.

10. Write a pair of equal angles for each of the triangles *AOC*, *FOG*, *QOR*.

(In 11—14, say which angle is greater, giving the reason fully.)

11. *OEC* and *OCD*.

12. *OHX* and *OKH*.

13. *OST* and *OTE*.

14. *EAB* and *BCA* in Diagram IV.

15. In Diagram IV. *ABCDEF* has six equal sides; find six pairs of equal angles in the figure.

16. *O* is the centre of the circle *ABC* in the same; prove the angles *OAF* and *OEF* are equal.

36. THE MUTUAL RELATIONS OF PROPS. I.—VI.

The leading proposition of this section is the fourth, in which, under certain circumstances, two triangles are shown to be equal in all respects; and it is the first case of the kind. The next two are deductions from it; and the first three in the book are required in order to assist in making these deductions.

If, then, superposition proofs were adopted for V. and VI., the problems I.—III. might be dispensed with so far as Section I. is concerned.

Examination XVIII.

1. Give the definitions of 'theorem,' 'hypothesis,' and 'thesis' or 'conclusion.' 2. Separate hypothesis from thesis in the following theorems:—(*a*) If *ABCD* is a square, its diagonal will bisect it. (*b*) If *ABC* is a circle, its diameters are all equal. (*c*) If two adjacent sides of one oblong be equal to two adjacent sides of another, the two oblongs shall be equal. 3. Separate thesis from hypothesis in the enunciation of Proposition IV. 4. Show that the hypothesis is threefold. 5. Show that the thesis is fourfold, enumerating the four portions of it. 6. State the relation in the hypothesis of Prop. IV. between the equal angles and the two pairs of equal sides. 7. If the triangles *ABC*, *DEF* have *AB*, *AC* equal to *DE*, *DF*, each to each, and also angles *ABC*, *DEF* equal, is this an example of the hypothesis of Prop. IV.? Give your reason. 8. Given, in the triangles *ABC*, *DEF*, that *BC*, *EF* are equal, and also *CA*, *FD*; what further is necessary to enable us to say that *AB*, *DE* are equal? 9. Separate hypothesis from thesis for Prop. V. 10. Count, in the same proposition, the lines of printing which form the construction. 11. Define 'Converse' Theorems. 12. Write the theorems which would be respectively converse to—(*a*) If the opposite sides of a figure be equal, they shall be parallel. (*b*) If *A* is *B*, then *X* shall be *Y*. (*c*) Proposition V. of Euclid, first part. 13. If *ABC* be a triangle, which side subtends the angle *A*, which the angle *B*, and which *C*? 14. Analyse the particular enunciation of Proposition VI. into *hypothesis* and *thesis*. 15. State the false or trial hypothesis of the reasoning, the false conclusion. What part of the particular enunciation does the false hypothesis contradict? 16. Show that Prop. VI. is the second of two converse theorems proved by Euclid. 17. Explain 'problem'; and say, in regard to each of the following, whether it is a theorem or a problem:—(*a*) Make a square equal to a given oblong. (*b*) If one angle of a square be right, all are so. (*c*) The centre bisects all diameters of a circle. 18. Show that an application of Prop. III. in Euclid is, strictly speaking, an application of all the first three propositions.

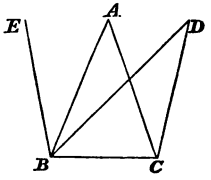
SECTION II. PROPOSITIONS VII.—XII.

Second case of equality of Two Triangles, with applications to Problems on Bisectors and Perpendiculars.

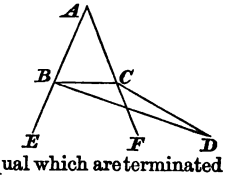
EXERCISE XI.

(*Note*—The word 'vertex' is used in Euclid for an angle opposite to the base of a triangle. In 1—5, Diagram IV. is referred to.)

1. Mention four vertices of triangles which are upon the base AC.
2. Also of all the triangles upon the base CD.
3. Name three triangles which are upon the base AE, and one one side of it.
4. Name the sides of the triangles upon base CD which are terminated in the extremity C; also those terminated in D.
5. Name two triangles upon base CE which have the vertex of one within the other.
6. In the figure to I. 5, if BG, CF interest in H, name all the triangles upon the base BC.
7. In the same figure, name two triangles on one side of BC which have the vertex of each outside the other.
8. Also two, of which the vertex of one lies on a side of the other.
9. In the annexed figure, if ABC be an isosceles triangle, prove that the angle EBC is greater than ACB.



10. Also that angle DBC is less than ACB.
11. Also that angle DCB is greater than DBC.
12. Prove that ABC and DBC in the figure cannot both be isosceles triangles.
13. In the annexed figure, having A and D on opposite sides of BC, let ABC be isosceles; then prove that the angle BCD is greater than EBC.



14. Also that angle DBC is less than BCF.
15. If ABC and DBC be two triangles, and D be a point in the side AC of the first of them, show that these triangles cannot have those sides equal which are terminated in the extremity C of the base.

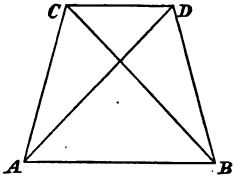
16. Also show that, if *CAB*, *CDB* are both equilateral triangles on opposite sides of *CB*, then the triangle *CAD* is isosceles.

37. Proposition VII.—Theorem.

Upon the same base and upon the same side, there cannot be two triangles which have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, upon the same base *AB*, and on the same side of it, let there be two triangles *ACB*, *ADB*,
which have not only their sides *CA*, *DA*, terminated in the extremity *A* of the base, equal to one another,
but *also* their sides *CB*, *DB*, terminated in *B*.
Join *CD*.

First, let the vertex of each triangle be without the other triangle.



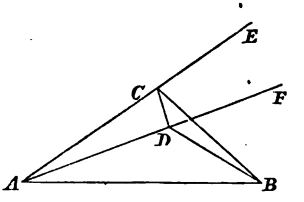
Because, in the triangle *ACD*, *AD* is equal to *AC*,
therefore the angle *ADC* is equal to the angle *ACD*; (I. 5)
but the angle *ACD* is greater than the angle *BCD*; (Ax. 9)
therefore *ADC* is also greater than *BCD*;
even more, then, is the angle BDC greater than BCD. (Ax. 9)

Again, in the triangle *BCD*, *BD* is equal to *BC*;
therefore the angle BDC is equal to the angle BCD; (I. 5)

Hence the angle *BDC* is both greater than, and equal to, the angle *BCD*;
which is impossible.

Secondly, let the vertex D of the triangle ADB fall within the triangle ACB.

Produce AC and AD to E and F.



Then, since AD is equal to AC in the triangle ACD,
the angles FDC, ECD upon the other side of CD are equal; (I. 5)
but the angle ECD is greater than the angle BCD; (Ax. 9)
therefore also FDC is greater than BCD;

Even more, then, is the angle BDC greater than BCD. (Ax. 9)

Again, in the triangle BCD, BD is equal to BC,

therefore the angle BDC is equal to BCD. (I. 5)

Hence the angle BDC is both greater than, and equal to, the angle BCD;
which is impossible.

Thirdly, let the vertex of one triangle be upon a side of the other.

This case needs no demonstration.

Therefore, upon the same base, &c. Q. E. D.

Exercise XII.

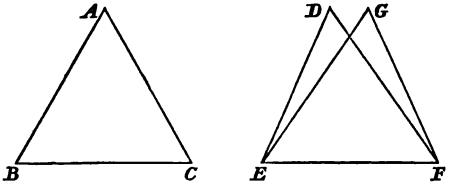
1. Does Prop. VII. deny that CA, DA in the figure can be equal? Does it deny that CB and DB can be equal? State precisely what it does deny.
2. Prove by Prop. VII. that only one equilateral triangle can be described on the same side of a given straight line as base.
3. Two circles cannot intersect more than once on the same side of the straight line which joins their centres. (If possible, let them do so and join the intersections to the centres.)
4. If a short proof were given of the third case of Prop. VII., on what axiom should it be based?

38. Proposition VIII.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also their bases equal, then they shall have those angles equal which are opposite to their bases.

Let ABC, DEF be two triangles, having the two sides AB, AC and the base BC
equal to the two sides DE, DF and the base EF, each to each, namely, AB to DE, AC to DE, and BC to EF.

Then the angles BAC, EDF, opposite to BC and EF, shall be equal.



For, if the triangle ABC be applied to the triangle DEF;
so that B may be upon E, and BC upon EF;
because BC is equal to EF, (Hyp.)
therefore C shall coincide with F;
wherefore BC coincides with EF. (Ax. 10)

Hence BA and AC shall also coincide with ED and DF.
For, *if possible*, let them have a different situation, as EG, GF.
Then, upon the same base, and upon the same side of it, there will be two triangles,
which have their sides DE, GE, terminating in E, equal,
and also DF, GF, which terminate in F;
which is impossible. (I. 7)

Therefore the sides BA, AC will fall upon ED, DF;
so that the angle BAC will coincide with EDF.
Therefore the angles BAC, EDF are equal. (Ax. 8)

Hence, if two triangles have two sides, &c. Q. E. D.

39. Equalities in Triangles.

After studying Proposition VIII., the learner should compare the enunciations of Props. IV. and VIII. very carefully; and notice that in each hypothesis we have *three parts* of one triangle given equal to *three parts* of another triangle, each to each. In Prop. IV., however, we draw *four* separate conclusions respecting the two triangles, while in Prop. VIII. we draw *one* conclusion only.

This latter fact may possibly be due to some oversight on the part of Euclid; for it is just as easy to continue the reasoning in Prop. VIII. until we arrive at four equalities, as it is in Prop. IV.; and, if we were to do so, it would prove of frequent use afterwards. Moreover, the continuation of the reasoning which is given in the one case, would precisely answer for the other.

The three conclusions of IV., which might have been, but are not, given in VIII. also, are—

(1) The triangles are equal themselves; that is, are equal in area;
(2) The angles opposite the first pair of equal *sides* are equal;
(3) The angles opposite the second pair of equal *sides* are equal.

We have, then, to remember that, when Euclid requires to prove one of these things, he falls back upon Prop. IV., even when he has just been using Prop. VIII. In a similar needless manner, Prop. IV. is used after XXVI. in XXXIV.

It should be noticed, however, that *three* of the four conclusions we have referred to above may be drawn from Prop. VIII., as it stands in Euclid, by *three successive* applications of it. This arises from the fact that any side we choose may be taken as base. Thus, if the triangles ABC, DEF have the three sides of AB, BC, CA respectively equal to the three DE, EF, FD; then, by I. 8, calling BC, EF the bases, we have angles A and D equal; calling CA, FD the bases, we have angles B and E equal; calling AB, DE the bases, we have angles C and F equal.

Exercise XIII.

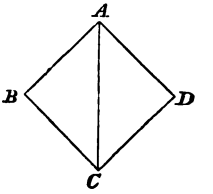
1. Given that the sides AB, BC, CA of the triangle ABC are respectively equal to DF, FE, ED of the triangle DEF; write out the three pairs of equal angles.

2. Given that BA, AC, CB are equal to DE, EF, FD, each to each; write out three pairs of equal angles.

3. Also, when PQR, XYZ are two triangles, having PQ, QR, RP respectively equal to YZ, ZX, XY, write out pairs of equal angles.

4. The diagonal of a rhombus bisects the angles through which it passes, and also bisects the figure itself.

The solution to this exercise is given in order to show how Euclid would apply in one proof the Eighth and Fourth Propositions respectively.



Let $ABCD$ be a rhombus. Join AC.

Then AC shall bisect the angles BAD, BCD and also the rhombus $ABCD$.

Because BA, AC are equal to DA, AC, each to each,

and the bases BC, DC are equal;

therefore the angles BAC, DAC are equal; (I. 8)

that is, AC bisects the angle BAD.

In the same way, we may show that AC bisects the angle BCD.

Again, BA, AC are equal to DA, AC, each to each,

and the included angle BAC is equal to the included angle DAC;

therefore the triangle BAC is equal to the triangle DAC; (I. 4)

that is, AC bisects the rhombus $ABCD$.

5. If two adjacent angles of a rhombus are each right, prove that the figure is a square.

6. If two isosceles triangles be upon the same base, but on opposite sides of it, the straight line joining the vertices will bisect each vertical angle (or the angle at each vertex).

40. Proposition IX.—Problem.

To bisect a given rectilineal angle.

Let BAC be the given rectilineal angle.

It is required to bisect BAC.

In AB take any point D;

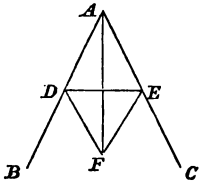
from AC cut off AE equal to AD; (I. 3)

join DE; and upon DE, on the side of it remote from A,

describe an equilateral triangle DEF; (I. 1)

join AF.

Then the straight line AF shall bisect the angle BAC.



In the triangles DAF, EAF,
AD is equal to AE, and AF common, (Constr.)
therefore the two sides DA, AF are equal to the two EA, AF, each to each;
also, the base DF is equal to the base EF. (Constr.)
Therefore the angle DAF is equal to the angle EAF; (I. 8)
that is, the angle BAC is bisected by the straight line AF.
Q. E. F.

EXERCISE XIV.

1. Apply Prop. VIII. to the figure of IX. so as to prove that the angles ADF, AEF are equal.

2. If a quadrilateral have two adjacent sides equal to one another, and also the other two sides equal, prove that one of its diagonals bisects the angles through which it passes.

3. Then prove that this diagonal bisects the other diagonal.

4. If a quadrilateral have the two sides which contain one angle equal to one another, and a diagonal bisecting the same angle, prove that the remaining sides of the quadrilateral will be equal to one another.

41. PROPOSITION X.—PROBLEM.

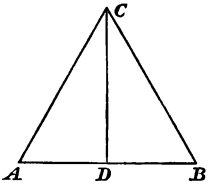
To bisect a given finite straight line.

Let AB be the given straight line.

It is required to divide AB into two equal parts.

Upon AB describe the equilateral triangle ABC; (I. 1)
and bisect the angle ACB by the straight line CD, (I. 9)
meeting AB in the point D.

Then AB shall be bisected in point D.



Because, in the two triangles ACD, BCD,
AC is equal to CB, and CD common; (Constr.)
the two sides AC, CD are equal to BC, CD, each to each;
and the angle ACD is equal to BCD. (Constr.)

43. Proposition XII.—Problem.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it.

It is required to draw from C a perpendicular to AB.

Take any point D upon the other side of AB;

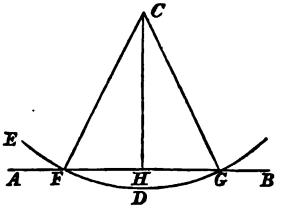
with centre C, at the distance CD, describe the circle EFG, meeting AB in F and G; (Post. 3)

bisect FG in H, (I. 10)

and join CH.

Then CH shall be drawn perpendicular to AB.

Join CF and CG.



Then, in the triangles FHC, GHC,

FH is equal to GH, and HC common; (Constr.)

therefore the two sides FH, HC are equal to the two GH, HC, each to each;

and the base CF is equal to the base CG. (Def. 15)

Therefore the angle FHC is equal to GHC; (I. 8)

and these are adjacent angles.

Hence each of the angles FHC, GHC is a right angle,

and the straight line CH is perpendicular to AB. (Def. 10)

Wherefore, from the given point C, a perpendicular CH has been drawn to the given straight line AB. Q. E. F.

Exercise XVII.—(Recapitulatory.)

1. $ABCD$ is a circle, whose centre is O, and the chords AB, BC, CD are respectively 3 ft., 4 ft., 3 ft. long; prove that two of the angles AOB, BOC, COD are equal.

2. Deduce a method of testing the equality of two angles at one point.

3. Make a straight line equal to one-and-a-half times a given straight line.

4. At a given point in a given straight line, construct an angle equal to a right angle.

5. At a given point in a given straight line, construct an angle equal to half a right angle.

6. Make a right-angled triangle, having each of the sides containing the right angle equal to a given straight line.

7. Construct a right-angled triangle, having one of the sides containing the right angle equal to a given straight line, and the other half as long.

44. Parts of a Proposition.

We have already referred to the enunciation, construction, and demonstration of a proposition. There is but little to add in order to complete, sufficiently for us, the analysis of a proposition.

The student should notice that each enunciation printed in italics is followed by a re-statement of itself referred by means of letters to a specific figure. These two statements are distinguished from each other by calling the former the *general* enunciation, and the other the *particular* enunciation; that is, the enunciation which applies to the particular diagram which we find it convenient to construct for the purposes of the proof.

Thus, in a proposition, we have as its leading divisions:—

(1) General enunciation $\left\{\begin{matrix}\text{hypothesis}\\ \text{conclusion}\end{matrix}\right\}$ or $\left\{\begin{matrix}\text{datum}\\ \text{quaesitum}\end{matrix}\right\}$;

(2) Particular enunciation, with similar parts;

(3) Construction; and

(4) Proof or demonstration.

45. Mutual Relations of Propositions VII.—XII.

Of these, the leading proposition is VIII.; which is the second case of equality, in all respects, of two triangles under given conditions.

Proposition VII. is accessory to it, and is not elsewhere applied in Euclid's Elements; while IX.—XII. are problems depending upon it for their solution.

Examination XIX.

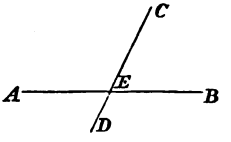
1. Describe the relation in which Prop. VII. stands to Prop. VIII. 2. Describe the relation between Prop. VIII. and the four problems which follow it. 3. Show that Prop. VIII. is applied in *all* these four problems. 4. Contrast the relation of problems to theorems in Props. I.—VI. with the relation of problems to theorems in Props. VII.—XII. 5. Distinguish

hypothesis from conclusion in the enunciation of Prop. VII. 6. State the nature of the proof of the same proposition. 7. What is the purpose of using the word "finite" in the enunciation of Prop. X.? 8. Analyse Prop. XI. into datum, quaesitum, particular enunciation, construction, and proof. 9. In the figure of Prop. XI., suppose the triangle *DEF* were made isosceles, what alteration would be required in the proof? 10. In the construction of XII., suppose *D* taken so far beyond *AB* that the point *A* falls *within* the circle *EFG*; how would that affect the construction? 11. Two lines are drawn in the same which are not necessary for obtaining the perpendicular. Name them, and say why they are drawn. 12. Define a *chord* of a circle.

SECTION III. PROPOSITIONS XIII.—XVII.

On the Angles which Two Intersecting Straight Lines make with one another or with a Third.

46. Exercise XVIII.



1. Name four pairs of adjacent angles made at the point *E* by the straight lines *AB*, *CD*.

2. Which of these pairs of adjacent angles is on one side of *AB* and above it? Which is below the same line? Which is on the same side of *CD* as *B*?

3. If *AEC* and *BEC* be equal, what may we call each of them? (Quote the definition which applies.)

4. If *AEC*, *BEC* be not equal, and a straight line be drawn upwards from *E* at right angles to *AB*, show that it cannot coincide with *EC*.

5. Hence infer that, if *AEC*, *BEC* be unequal, one is acute, the other obtuse.

6. In the second figure to Prop. XIII., *CBE* is a right angle. By how much is the angle *CBA* greater than a right angle? How much is the angle *DBA* less than a right angle?

7. In the same figure, take the angle *EBA* from the angle *CBA* and add it to the angle *ABD*; what two angles have thus been constructed out of the two angles *CBA*, *ABD*?

Draw an inference with regard to the magnitude of the sum of any two adjacent angles.

8. Take the three angles of the same figure, and show that by adding two of them together we may form two right angles out of them.

9. Show that the two angles *CBA*, *ABD* may be constructed out of the same three angles.

10. By means of the two constructions of Questions 8 and 9, show that the angles *CBA*, *ABD* are together equal to two right angles.

47. Proposition XIII.—Theorem.

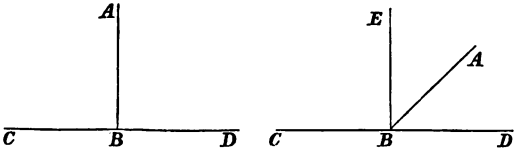
The angles which one straight line makes with another, upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line *AB* make with *CD*, upon one side of *CD*, the angles *CBA*, *ABD*.

Then these shall either be two right angles, or shall be together equal to two right angles.

For, if the angle *CBA* be equal to the angle *ABD*,
each of them is a right angle. (Def. 10)

But, if the angles *CBA*, *ABD* be not equal,
from the point *B* draw *BE* at right angles to *CD*; (I. 11)
so that the angles *CBE*, *EBD* are two right angles.



The whole angle *EBD* is equal to the two *EBA*, *ABD*;
add the angle *CBE* to each of these equals;
then the angles CBE, EBD are equal to the three angles CBE, EBA, ABD. (Ax. 2)

Again, the angle *CBA* is equal to the two *CBE*, *EBA*;
to each of these equals add the angle *ABD*;
then the angles CBA, ABD are equal to the three angles CBE, EBA, ABD.

But the angles *CBE*, *EBD* have been proved equal to the same three angles;

and things which are equal to the same are equal to one another;

therefore the angles CBA, ABD are equal to the angles CBE, EBD. (Ax. 1)

But *CBE*, *EBD* are two right angles. (Constr.)

Therefore the angles *CBA*, *ABD* are together *equal* to two right angles.

Wherefore the angles which one straight line, &c.

Q. E. D.

48. Supplementary Angles.

Def.—When two angles, taken together, are equal to two right angles, either of them is called the supplement of the other.

Such angles are called 'supplementary.' Proposition XIII. treats of the most important case of supplementary angles; namely, those which are adjacent.

It may be observed that Prop. XIII. is proved on the simplest principles we can employ; namely, the earlier axioms. There is nothing in its *proof*, therefore, to prevent its being made the first proposition in the subject. The *construction*, however, will supply the reason for its present position (after Prop. XI.); for Euclid never adds a line of construction whose method has not previously been established. Some writers on Geometry consider Euclid's caution in this regard to be carried farther than is necessary for safe argument.

Exercise XIX.

1. In the application of Axiom 1, in the proof, state the magnitude which is used for successive comparison with two others. Also state distinctly these other magnitudes.

2. In the figure of Ex. XVIII. 1, divide the angles at *E* into *two* pairs of adjacent angles; and deduce from Prop XIII. the magnitude of all the four taken together.

3. In the figure of Prop. XIV., if *CB*, *BE* form one straight line, what may the angles *CBA*, *ABE* be called?

4. Given, in the figure to Prop. XV., that *AB*, *CD* are two straight lines; state the magnitude of the two angles *CEA*, *AED* taken together.

5. Show that the sum of those angles is equal to the sum of the angles *BED*, *DEA*.

6. Applying Axiom 3 to these equals, show that two of the angles mentioned are equal.

7. By a similar method, show that the angles *AED*, *BEC* are equal.

8. If two straight lines intersect, show that two adjacent angles thus formed cannot be both acute, nor can they be both obtuse.

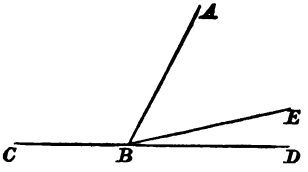
49. Proposition XIV.—Theorem.

If at a point in a straight line two other straight lines upon the opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point *B* in the straight line *AB*,
let the straight lines *BC*, *BD*, upon opposite sides of *AB*,
make the adjacent angles *ABC*, *ABD* together equal to two right angles.

Then BD shall be in the same straight line with BC.

For, if *BD* be not in one straight line with *BC*,
let *BE* be in a straight line with *BC*,



Then, because *CBE* is one straight line,
therefore the angles *CBA*, *ABE* are together equal to two right angles; (I. 13)
but the angles *CBA*, *ABD* are equal to two right angles; (Hyp.)
therefore CBA, *ABE are together equal to CBA*, *ABD.*
Take the common angle *CBA* from these equals;
then the remaining angles *ABE*, *ABD* are equal; (Ax. 3)
the less equal to the greater; which is impossible.

Therefore BE is not in the same straight line with BC.

In the same manner, it may be demonstrated that
no other can be in the same straight line with it except BD;
therefore *BD* is in the same straight line with *CB*.

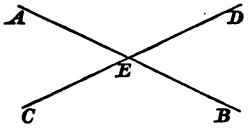
Wherefore, if at a point, &c. Q. E. D.

50. Proposition XV.—Theorem.

If two straight lines cut one another, the opposite vertical angles shall be equal.

Let the two straight lines *AB*, *CD* cut one another in the point *E*.

Then the angle AEC shall be equal to the angle BED, and the angle AED to the angle BEC.



Because *AE* makes with *CD* the adjacent angles *CEA*, *AED*, these angles are together equal to two right angles. (I. 13)

Again, because *DE* makes with *AB* the adjacent angles *AED*, *DEB*,

these angles also are together equal to two right angles. (I. 13)

But *CEA*, *AED* have been shown to be together equal to two right angles.

Therefore CEA, AED are together equal to AED, DEB.

Take away the common angle *AED*;

then the remaining angle CEA is equal to the remaining angle DEB. (Ax. 3)

In the same manner, it may be demonstrated *that the angle AED is equal to the angle BEC.*

Wherefore, if two straight lines cut one another, &c.

Q. E. D.

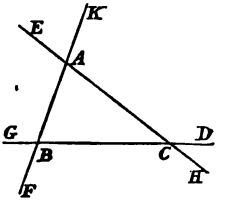
Cor. 1.—From the mode of demonstration, it is manifest that, if two straight lines cut each other, the angles which they make at the point where they cut are together equal to four right angles.

Cor. 2.—And, consequently, that all the angles made by any number of straight lines, meeting in one point, are together equal to four right angles.

Note.—These two important corollaries are usually given here; but

they are really direct inferences from Prop. XIII.; and their proper place would be immediately after that proposition.

51. INTERIOR AND EXTERIOR ANGLES OF TRIANGLES.



In the accompanying diagram, the sides of the triangle ABC will be found to be each produced both ways. The number of the angles at the points A, B, and C is thereby increased fourfold. The angles DCH, EAK, and FBG will not come under present notice. Of the rest, we usually speak of three as the *interior* angles of the triangle; and these are, of course, CAB, ABC, BCA. The remaining six are called *exterior* angles of the triangle, and each of these is contained by one *side* of the triangle and another *side produced*. They are in two sets; one set being formed by producing the sides successively one way, as BC to D, CA to E, and AB to F; the other by producing the sides successively the opposite way, as CB to G, AC to H, BA to K. When we speak of *three* exterior angles, we mean *one* of these sets only, and *either* may be taken.

Take any exterior angle as ACD; it evidently has one interior angle *adjacent* to it. The remaining interior angles are considered as its *opposite* interior angles; thus, the opposite interior angles to ACD are CAB, ABC.

EXERCISE XX.

1. Show that the exterior angles of the triangle ABC consist of three pairs of equal angles.

2. Hence deduce that the sum of the angles made by producing the sides successively in one direction, are respectively equal to those made by producing the sides successively in the other direction.

3. Give the interior angles respectively adjacent to BCH, CAK, ABG in the foregoing figure.

4. Give the pairs of interior angles respectively opposite to the same angles.

5. Draw two straight lines AB, CD bisecting one another at E, and join AC, BD; then show that the triangles AEC, BED will have three parts of the one respectively equal to three parts of the other.

6. Draw the inferences which follow by the application of Prop. IV.

7. In the figure to Prop. XVI., given that AC, BF bisect one another in E, prove that the triangles ABE, CEF are equal in all respects.

8. Assuming this result, show that the angle A is less than the angle ACD.

9. Deduce that the angle BCG is greater than the angle A.

10. In the same figure, prove that the angle F is less than the angle ABC.

52. Proposition XVI.—Theorem.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

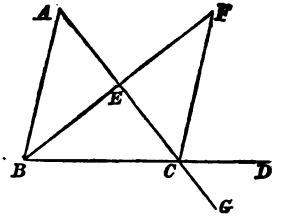
Let ABC be a triangle whose side BC is produced to D.

Then the exterior angle ACD shall be greater than either of the interior opposite angles CBA or BAC.

Bisect AC in E; (I. 10)

join BE, and produce BE to F, making EF equal to BE; (I. 3)

and join FC.



Because AE is equal to CE, and EB to EF;

therefore in the two triangles EAB, ECF,

the two sides AE, EB are equal to the two CE, EF, each to each,

and the angle AEB is equal to the angle CEF,

because they are opposite vertical angles. (I. 15)

Therefore the angles which are opposite to the equal sides EB, EF are also equal;

namely, the angles EAB and ECF. (I. 4)

But the angle ACD is greater than the angle ECF;

therefore ACD is also greater than the angle CAB.

In the same manner, if the side BC be bisected,

and AC be produced to G;

it may be demonstrated that the angle BCG,

and therefore, also, the angle ACD, (I. 15)

is greater than the angle ABC.

Therefore, if one side, &c. Q. E. D.

Exercise XXI.

1. Make the construction referred to in the latter part of the proof, and give the full proof that the angle BCG is greater than the angle ABC.

2. In the figure to Prop. XVI., show that the angle A is less than the angle BEC; and that the angle FEC is less than FCG.

3. Prove that the angle FEC is greater than the angle GCD.

4. Prove that the angle BEC is intermediate in value between the angles A and ACD.

5. In the figure of Prop. XVIII., which is the greater angle in each of these pairs: A and BDC, C and BDA?

6. In the same figure, prove that the angles A and C are together less than the adjacent angles at the point D taken together.

7. Thence infer that the angles A and C together cannot be equal to the sum of two right angles.

8. Prove the same with regard to the angles B and C of any triangle ABC, by joining A to any point in BC.

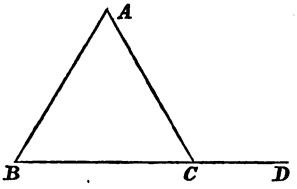
53. Proposition XVII.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Then any two of its angles together shall be less than two right angles.

Produce any side BC to D.



Then ACD is an exterior angle of the triangle ABC,
and ABC is an interior angle opposite to it;
therefore the angle ACD is greater than ABC. (I. 16)
To each of these unequals add the angle ACB;
therefore the angles ACD, ACB are greater than ABC, ACB.
But ACD and ACB are equal to two right angles; (I. 13)
therefore ABC and BCA are together less than two right angles.

In like manner, it may be demonstrated that the angles BCA, CAB are less than two right angles; as also the angles CAB, ABC.

Therefore, any two angles, &c. Q. E. D.

Exercise XXII.

1. Examine whether the proof would hold good, with the same construction exactly, if the angle BAC were substituted for ABC.

2. Also give the proof, producing CB instead of BC, employing the angle ACB and denoting it by C.

3. Show that, if one angle of a triangle be right, each of the others must be acute.

4. Show that not more than one perpendicular can be drawn to a straight line from a point without it.

5. Show that only one angle of a triangle can be obtuse.

6. Show that the three interior angles of a triangle are together less than three right angles.

7. Show that the three exterior angles of a triangle made by producing the sides successively in the same direction are together greater than three right angles.

8. If a right-angled triangle be isosceles, the right angle is not subtended by one of the equal sides.

54. "The Twelfth Axiom" and its Converse.

It is advisable to consider what we have given as Axiom 12 (Art. 21) at this stage, or at least not earlier; for hitherto we have probably not been in a position to see clearly the nature of the assertion which it contains; and, certainly, it has not so far been a true axiom. But our knowledge of the circumstances attending angles formed by one straight line meeting two others has now grown much greater and more precise through the study of the last few propositions. It need not be surprising therefore if, on now examining the statement of Ax. 12, we find it come before us under quite a new light.

It will be better to make this examination of the axiom by means of some of the ensuing questions (18-20). One fact which we shall arrive at, however, is important enough to deserve special statement here; namely, that *Prop. XVII. and Axiom* 12 *are converse propositions.*

In the exercise last given, we have already called attention to another point of interest, which also is worthy of explicit mention.

On looking carefully at the definitions of a right-angled and an oblique-angled triangle (Art. 12), we observe that Euclid must have had Prop.

XVII. or else XXXII. in view when framing them. For this reason, therefore, these definitions would come most naturally at the present stage.

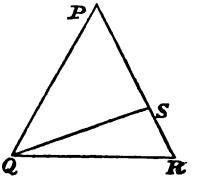
Examination XX.

1. Briefly analyse Prop. XIII. by stating how many lines of print are occupied by the *general enunciation*, *particular enunciation*, *construction*, and *proof*. In which of these are there two parts? 2. Is the proof direct or indirect? 3. What is the earliest position we may assign on Euclid's principles to Prop. XIII.? 4. State the angles in the second figure of XIII. which are respectively supplements to ABD, EBC, CBA. 5. When are two supplementary angles equal? What angle is supplementary to one-half of a right angle? 6. What proposition may be enunciated thus: If two straight lines intersect, any angle between them is supplementary to either of the angles adjacent to it. 7. What is the nature of the demonstration of Prop. XIV.? 8. State the false hypothesis and false conclusion of its reasoning. 9. Analyse the particular enunciation into hypothesis and thesis. 10. What proposition proves that, if two adjacent angles be supplementary, their outer bounding lines, when produced indefinitely, coincide? 11. Enunciate the converse proposition to XIV. 12. In a figure similar to that of Prop. XIV., suppose that ABE is equal to nine-tenths of a right angle, and ABD to eleven-tenths of a right angle. Show that the two straight lines BE, BD make with AB, at a point B in AB, two angles which are together equal to two right angles. State a condition of the enunciation of Prop. XIV. which these two lines do not fulfil. 13. Which proposition is equivalent to this:—If two straight lines which form an angle be both produced backward through their point of meeting, the new angle thus formed between the parts produced is equal to the original one. 14. Assuming that angles which have the same supplement are equal, deduce Prop. XV. 15. Enunciate a proposition which shall be converse to Prop. XV. 16. Define an exterior angle of a triangle. 17. Construct the figure of Prop. XVI. (1 by taking CB produced instead of BC, (2) taking AB produced instead of BC. 18. Which proposition may be enunciated thus:—If two intersecting straight lines are cut by a third, the interior angles which the latter makes with the two former on that side on which they meet, shall be together less than two right angles. 19. Give the hypothesis and the thesis of this proposition. 20. Give the hypothesis and the thesis of the Twelfth Axiom, and compare them with those of the last question. Deduce the relation which exists between Prop. XVII. and the Twelfth Axiom.

SECTION IV. PROPOSITIONS XVIII.—XXV.

Inequalities of Parts of Triangles, and Accessory Problems.

55. Exercise XXIII.



In the annexed diagram, PQR is an equilateral triangle; it is referred to in 1—7.

1. Name an exterior angle of the triangle PQS, and also of the triangle QRS.

2. Name the angle which is greater than angle P, by I. 16.

3. Also that which is greater than angle R.

4. Find two angles in the figure which are less than R, and prove them so.

5. From Questions 3 and 4 deduce which is the greater angle, PSQ or PQS.

6. Show that in magnitude angle P is intermediate between angles SQR and QSR.

7. Show that angle P is less than either angle at S.

8. Given a triangle PQS having the side PS less than the side PQ, show that the angle opposite PS is less than the angle opposite PQ.

(For construction, produce PS to R, making PR equal to PQ, and join QR.)

9. In the figure to Prop. XVIII., given AD equal to AB, prove that the angle ADB is intermediate in magnitude between the angles ABC and C.

10. If, in the same figure, the angle C is greater than DBC, can we say that DB and DC are equal? If not, give the reason fully, with reference.

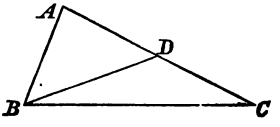
56. Proposition XVIII.—Theorem.

If a triangle have two sides unequal, the angle which is opposite the greater side is greater than the angle opposite the less.

Let ABC be a triangle,

of which the side AC is greater than the side AB.

Then the angle ABC shall be greater than the angle ACB.



From AC the greater cut off AD equal to AB the less, (I. 3) and join BD.

Then, because AD is equal to AB in the triangle ABD, the angle ABD is equal to the angle ADB; (I. 5)

and ADB is an exterior angle of the triangle BCD;

therefore it is greater than the interior opposite angle DCB; (I. 16)

therefore, also, the angle ABD is greater than DCB;

even more, then, is the angle ABC greater than the angle ACB.

Therefore, if a triangle have, &c. Q. E. D.

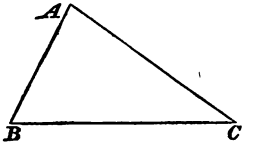
57. Proposition XIX.—Theorem.

If a triangle have two angles unequal, the side which is opposite the greater angle is greater than the side which is opposite the less.

Let ABC be a triangle,

of which the angle ABC is greater than the angle ACB,

Then the side AC shall be greater than the side AB.



For, if AC be not greater than AB,

AC must either be equal to or less than AB.

If AC were equal to AB,

then the angles ABC and ACB would be equal; (I. 5)

but they are not; (Hyp.)

therefore AC is not equal to AB.

Again, if AC were less than AB,

then the angle ABC would be less than ACB; (I. 18)

but it is not; (Hyp.)

therefore AC is not less than AB.

Hence, AC is neither equal to nor less than AB;

therefore AC is greater than AB.

Wherefore, if a triangle have, &c. Q. E. D.

Exercise XXIV.

1. In a scalene triangle, prove that each pair of angles is unequal.

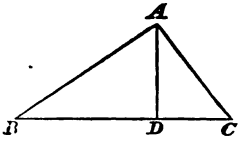
2. In a right-angled triangle, prove that the hypotenuse is the greatest side.

3. If two straight lines be drawn from a point to meet a straight line, one of which is perpendicular to it, show that this perpendicular must subtend an acute angle where the oblique meets the straight line to which it is drawn.

4. Show also that the perpendicular must be less than the oblique.

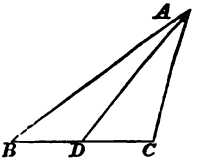
5. If there were two oblique lines on the same side of the perpendicular, show that the one nearer to the perpendicular is the lesser of the two.

6. In the annexed figure, AD is perpendicular to BC; prove that CD is less than AC, and BD less than AB.



7. Show from these inequalities that BC is less than BA and AC together.

8. In the next figure, AD bisects the angle BAC. Show that the angle CDA is greater than the angle CAD. (Apply I. 16.)



9. Draw an inference with regard to the comparative magnitudes of CD, CA.

10. Prove that BD is less than BA.

11. From Questions 9 and 10 deduce a method of proving that BC must be less than BA and AC together.

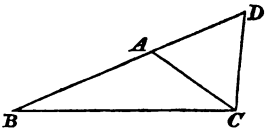
12. By bisecting the angle C, show that BC and CA are together greater than AB.

58. Proposition XX.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be any triangle.

Any two of its sides shall be together greater than the third; namely, BA, AC greater than BC; AB, BC greater than CA; and BC, CA greater than AB.



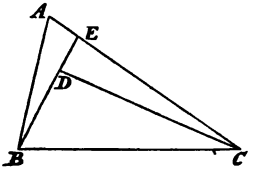
Produce BA to D, making AD equal to AC, (I. 3) and join DC.

Then, because AD is equal to AC,
the angle ACD is equal to the angle ADC; (I. 5)
but the angle BCD is greater than ACD;
therefore, also, BCD is greater than ADC;
that is, in the triangle BCD,
the angle BCD is greater than the angle BDC.
But the greater angle is subtended by the greater side; (I. 19)
therefore the side BD is greater than the side BC.
But BD is equal to BA and AC;
therefore BA and AC are together greater than BC.
In the same manner, it may be demonstrated
that the sides AB, BC are greater than CA,
and also that BC, CA are greater than AB.
Wherefore any two sides, &c. Q. E. D.

59. PROPOSITION XXI.—THEOREM.

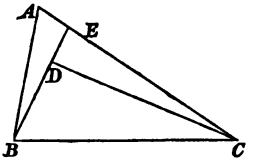
If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and D a point within it,
and from B, C, the ends of the side BC,
let two straight lines BD, CD be drawn to the point D.
Then BD and DC shall be less than BA and AC,
but shall contain an angle BDC greater than the angle BAC.
Produce BD to meet the side AC in E.



Since two sides of a triangle are greater than the third side, (I. 20)
therefore, in the triangle ABE,
the two sides BA, AE are greater than BE;
to each of these unequals add EC;
then the sides BA, AC are greater than BE, EC. (Ax. 4)

Again, in the triangle CED,
the two sides DE, EC are greater than DC; (I. 20)
add BD to each of these unequals;
then the sides BE, EC *are greater than* BD, DC. (Ax. 4)
But it has been shown that BA, AC are greater than BE, EC;
even more, then, are BA, AC greater than BD, DC.



Again, because BDC is an exterior angle of the triangle CDE,
it is greater than the interior opposite angle DEC. (I. 16)
And, because DEC is an exterior angle of the triangle BAE,
it is greater than the interior opposite angle BAC.
But it has been shown,
that the angle BDC is greater than the angle DEC;
even more, then, is the angle BDC greater than BAC.
Therefore, if from the ends, &c. Q. E. D.

Exercise XXV.

1. Give the proof of XX. that BA and AC are greater than BC, by producing CA instead of BA.

2. Give the proof in full that BC, CA are greater than BA.

3. Twice the longest side of a triangle must be less than the sum of the three sides.

4. Prove Prop. XXI. by producing CD to meet AB in E in the construction.

5. Prove that which relates to vertical angles, by means of joining AD, and then producing AD.

6. In an isosceles triangle, show that the base cannot be twice as great as one of the equal sides.

7. Given three straight lines respectively equal to the sides of a triangle; if the first be half of the second, show that the third must be greater than the first.

8. Given three pieces of straight wire, which are to be placed upon a table in the form of a triangle; if this cannot be done, what will be the cause?

60. Proposition XXII.—Problem.

To make a triangle, of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

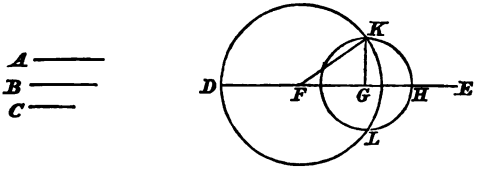
Let A, B, C be the three given straight lines,
of which any two whatever are greater than the third; (I. 20)
namely, A and B greater than C, A and C greater than B,
and B and C greater than A.

It is required to make a triangle of which the sides shall be equal to A, B, C, each to each.

Take a straight line DE,
terminated at the point D, but unlimited towards E.

Make DF equal to A, FG equal to B, and GH equal to C; (I. 3)
at the distance FD from the centre F describe the circle DKL;
at the distance GH from the centre G describe the circle HLK;
and join KF, KG.

Then the triangle KFG shall have its sides equal to the three given straight lines.



Because the point F is the centre of the circle DKL,
therefore FD is equal to FK; (Def. 15)
but FD is equal to the straight line A; (Hyp.)
therefore FK is equal to A. (Ax. 1)

Again, because G is the centre of the circle HLK,
therefore GH is equal to GK;
but GH is equal to C;
therefore, also, GK is equal to C.
And FG was made equal to B.
Therefore the three straight lines KF, FG, GK are respectively equal to the three A, B, C;

that is, the triangle KFG has been made having its three sides equal to the three straight lines A, B, C, each to each.

Q. E. F.

EXERCISE XXVI.

1. Construct the figure for the case in which the straight lines A and B are equal. 2. Also when A and C are equal.

3. Give construction and proof when B and C are equal.

4. Suppose a triangle ABC given; show how to make a triangle whose sides shall be respectively equal to the sides of ABC.

5. Show that the triangle constructed has also its angles respectively equal to those of ABC.

6. Given two straight lines A and B; show how to construct a rhombus having its diagonal equal to A, and each side equal to B.

61. PROPOSITION XXIII.—PROBLEM.

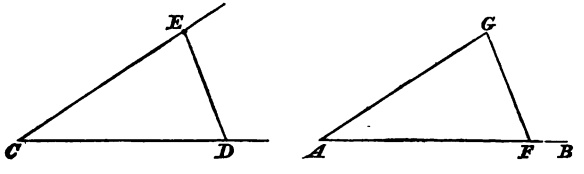
To draw a straight line from a given point in a given straight line, which shall make an angle with it equal to a given angle.

Let AB be the given straight line, A the given point in it, and DCE the given angle.

It is required to draw from A a straight line which shall make an angle with AB equal to DCE.

In CD, CE take any points D, E, and join DE; make the triangle AFG, the sides of which shall be equal to the three straight lines CE, CD, DE, (I. 22) so that AG is equal to CE, AF to CD, and FG to DE.

Then the angle FAG shall be equal to the angle DCE.



Because FA, AG are equal to DC, CE, each to each, and the base FG is equal to the base DE;

therefore the angle FAG is equal to the angle DCE. (I. 8)

Wherefore, at a given point A in the given line AB, the angle FAG is made equal to the given angle DCE.

Q. E. F.

Exercise XXVII.

For the figure, see that of the next Article, in which it is given that *AC and AD are equal.*

1. Which is the greater of the angles *DCA*, *DCB*? Quote the appropriate axiom.
2. Which is the greater of *ADC* and *CDB*?
3. Which is the greater, *ACD* or *BDC*?
4. Show that the angle *BDC* is greater than the angle *BCD*.
5. Infer from the previous result which is the greater, *BC* or *BD* in the triangle *BCD*.

62. On Lemmas.

Def.—A Lemma is a proposition *whose principal object is to assist in, or to simplify*, the demonstration of some other proposition; which, in general, it immediately precedes.

Proposition VII. may be considered a mere lemma to VIII.; since, but for its use in the latter proposition, we should probably not find it in Euclid's Elements at all. But it would not be correct to say, for example, of Prop. XXIII., that it is a lemma to XXIV.; because, although required in that proposition, it is also of frequent use elsewhere, and is, therefore, an independent proposition.

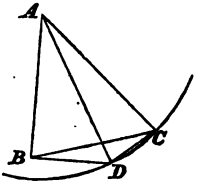
Prop. XXIV., as given in Euclid's Elements, really is in need of a lemma in order to clearly account for the method of its construction. One which will answer this purpose has been worked out in substance in the course of the preceding Exercise. It will now be formally given. Unless this lemma, or its equivalent, be taken into account, there should be three figures employed in the demonstration of Prop. XXIV.; in one of which *F* is upon *EG*, and in another above *EG*.

Lemma to Prop. XXIV.

If two triangles upon one base, and on one side of it, have the sides terminated in one extremity of the base equal, and each not less than the base, the vertex of either triangle will be without the other.

Given, on the same base *AB* and on the same side of it, the triangles *ABC*, *ABD*, placed as in annexed diagram, and which have *AC* equal to *AD* and each *not less* than *AB*;

then D and A will be on opposite sides of BC.



Proof.

With centre *A* and radius *AC* describe a circle.

Since *AC*, *AD* are equal, this will pass through *D*.

Since AB is either *equal* to the radius or *less* than it,
B will either be on the circumference or within the circle.
Hence triangle BCD has its base BC *within* the circumference ;
and its vertex D *on* the circumference.
Therefore D is on the other side of BC from the centre A.

63. Proposition XXIV.—Theorem.

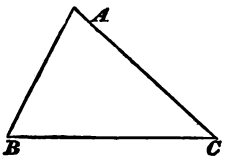
If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; then the base of that which has the greater angle shall be greater than the base of the other.

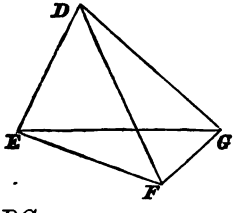
Let ABC, DEF be two triangles,
which have AB equal to DE, and AC to DF;
but the angle BAC greater than the angle EDF.

Then the base BC shall be greater than the base EF.

Of the two sides DE, DF, let DE be *not the greater*.

At the point D, in the straight line DE,
make the angle EDG equal to the angle BAC; (I. 23)
make DG equal to DF or AC (I. 3); and join EG, GF.





Then, in the triangles ABC and DEG,
because AB is equal to DE, and AC to DG,
and the angle BAC to the angle EDG;

therefore the base BC is equal to the base EG. (I. 4)

And because, in the triangle DFG, DG is equal to DF,
therefore the angle DFG is equal to the angle DGF; (I. 5)
but DGF is greater than the angle EGF, (Ax. 9)
therefore DFG is also greater than EGF;

even more, then, is the angle EFG greater than EGF.

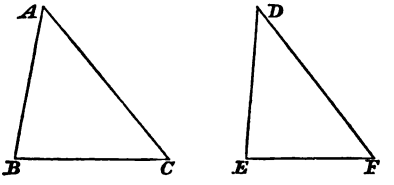
Now these are both angles of the triangle EFG;

and the greater angle is subtended by the greater side ; (I. 19)
therefore the side EG is greater than the side EF.
But *BC* was proved equal to *EG*,
therefore *BC* is greater than *EF*.
Wherefore, if two triangles, &c. Q. E. D.

64. PROPOSITION XXV.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other ; then the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides equal to them of the other.

Let *ABC*, *DEF* be two triangles,
which have *AB* equal to *DE*, and *AC* to *DF*;
but the base *BC* greater than the base *EF*.
Then the angle BAC shall be greater than the angle EDF.



For, if the angle *BAC* be not greater than the angle *EDF*, it must either be equal to it or less than it.

If *BAC* were equal to *EDF*,
the base *BC* would be equal to the base *EF*; (I. 4)
but it is not equal; (Hyp.)
therefore BAC is not equal to EDF.

Again, if *BAC* were less than *EDF*,
then *BC* would be less than *EF*; (I. 24)
but it is not less ; (Hyp.)
therefore BAC is not less than EDF.
And it has been shown that *BAC* is not equal to *EDF*;
therefore the angle *BAC* is greater than the angle *EDF*.
Wherefore, if two triangles, &c. Q. E. D.

Exercise XXVIII.

(For 1-13, refer to Diagram I.; for 14-19, to Diagram IV.)

1. Show, by I. 18, that the angle OCR is greater than the angle ORC.

2. Also, by I. 19, that in the triangle ERS the side ER must be greater than ES.

3. The angle EFT is obtuse; prove ET greater than either EF or FT.

4. Show that OG and GT together are greater than OP.

5. ER and RO are greater than ED and DO.

6. Which is the greater angle, EDO or ERO? Why?

7. Also EDC or ERC? Give the reason.

8. Show that EO, OT are equal to CO, OR, each to each.

If it be given that EOT is greater than COR, what is the inference with regard to CR and ET?

9. Given angle AOC greater than DOE, prove AC greater than DE.

10. Given angle POQ greater than QOR, prove PQ greater than QR.

11. Given angle EOF greater than GOF, prove ET greater than GT.

12. If CD be less than DE, show that the angle COD is less than DOE.

13. If ET be greater than ER, show that angle EOT is greater than EOR.

14. Find a straight line equal in length to AO and OC together.

15. Prove that AD is equal to EO and OC together.

16. Show that the angle OAE is less than AEC.

17. If BDF be equilateral, prove that the angle AFD is greater than the angle BDF.

18. If AFD be a right angle, show that DF is less than AO and OC together.

19. The sum of the sides of the figure $ABCDEF$ is greater than the sum of the sides of the triangle ACE.

Examination XXI.

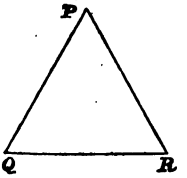
1. Give the hypothesis and thesis of Prop. XVIII. and also of XIX.; and state the relation of the one proposition to the other. 2. What are the respective methods of proof of XVIII. and of XIX.? 3. How many false hypotheses are tested in the latter? State these, and also the false conclusions deduced from them. 4. What proposition may be enunciated thus: If two triangles are upon one base, and have the vertex of one within the other, show that the two sides of the inner triangle are less than those of the outer, but the inner vertical angle is greater than the outer one. 5. Where in the previous propositions has it been proved that the sides of the inner triangle cannot be respectively equal to the sides of the outer? 6. In Prop. XXII. count the number of printed lines devoted to the general enunciation, particular enunciation, construction, and proof respectively. 7. Mention a previous proposition which is a

particular case of Prop. XXII. 8. What is a Lemma? State an example of a lemma among the propositions of Euclid which have been already given. 9. In the figure of XXIV., DG is taken equal to DF, and *not less* than DE. What is thus ensured with regard to the point F in the figure?

SECTION V.—PROPOSITION XXVI.

Third and Fourth Cases of Equality of Two Triangles.

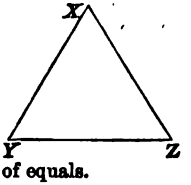
EXERCISE XXIX.



1. In the triangle PQR, which side is adjacent to both the angles Q, R?

2. Given that angles Q, R are equal to angles Y, Z, each to each, and the sides adjacent to those angles are also equal; write out these given parts of the two triangles as three pairs of equals.

3. Write out the remaining sides in pairs, the two in each pair being opposite to equal angles.

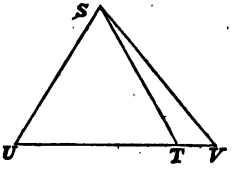


4. Given the angles Q, R equal to Y, Z, each to each, and the sides also equal which are *opposite* the *first* pair of equal angles; write these parts of the two triangles in three pairs of equals.

5. Write out the pair of third angles.

6. Given angles P, R equal to X, Z, each to each, and the sides also equal which are opposite the *second* pair of equal angles; write these in three pairs of equals.

7. Given in PQR, XYZ that PQ, QR, and angle Q, are respectively equal to XY, YZ, and angle Y; show that angles R and Z are equal.



8. In the annexed figure, show that the angle V is less than the angle STU.

9. Given that SU, UV, and the angle U, are equal to XY, YZ, and the angle Y; show that the angle STU is greater than the angle Z.

10. Given that SU, UV, and the angle U, are equal to PQ, QR, and the angle Q; show that the angle P is greater than the angle UST.

11. Given ST, TU, US respectively equal to PR, RQ, QP; determine which is the greater of the angles R and V.

65. Proposition XXVI.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles or the sides opposite to one pair of equal angles; then shall their other sides be equal, each to each, and also their third angles.

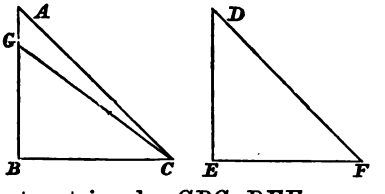
Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, namely, ABC to DEF, and BCA to EFD;
also one side equal to one side.

First, let those sides be equal which are adjacent to the equal angles, namely, BC to EF.

Then the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.

For, if AB be not equal to DE,
one of them must be greater than the other.

Let, *if possible*, AB be greater than DE;
make BG equal to ED (I. 3), and join GC.



Then, in the two triangles GBC, DEF,
because GB is equal to DE, and BC to EF; (Hyp.)
and the angle GBC is equal to the angle DEF;
therefore those angles are equal which are opposite the equal sides GB, DE, (I. 4)

that is, the angle GCB is equal to the angle DFE.

But DFE is equal to ACB.
Wherefore, also, GCB is equal to ACB,
the less equal to the greater, which is impossible;
therefore AB is *not* unequal to DE,

that is, AB is equal to DE.

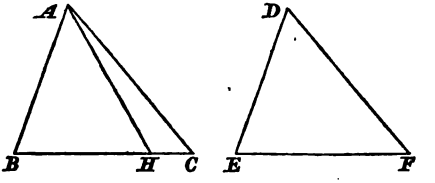
Hence, in the triangles *ABC*, *DEF*,
because *AB* is equal to *DE*, and *BC* to *EF*,
and the angle *ABC* to the angle *DEF*,
therefore the base AC is equal to the base DF, (I. 4)
and the third angle BAC to the third angle EDF.

Secondly, let sides in each triangle be equal which are opposite one pair of equal angles, namely, *AB* equal to *DE*.

Then, in this case likewise, the other sides shall be equal, namely AC to DF, and BC to EF,
and also the third angle BAC to the third angle EDF.

For, if *BC* be not equal to *EF*,
one of them must be greater than the other.

Let, *if possible*, *BC* be greater than *EF*,
make *BH* equal to *EF* (I. 3), and join *AH*.



Then, in the two triangles *ABH*, *DEF*,
because *AB* is equal to *DE*, and *BH* to *EF*,
and the angle *ABH* to the angle *DEF*; (Hyp.)
therefore the angles opposite to *AB*, *DE* are equal, (I. 4)
that is, the angle BHA is equal to the angle EFD.
But the angle *EFD* is equal to *BCA*; (Hyp.)
therefore *BHA* is equal to *BCA*, which is impossible. (I. 16)
Wherefore *BC* is *not* unequal to *EF*,
that is, BC is equal to EF.

Hence, in the triangles *ABC*, *DEF*,
because *AB* is equal to *DE*, and *BC* to *EF*, (Hyp.)
and the angle *ABC* is equal to the angle *DEF*,
therefore the base AC is equal to the base DF, (I. 4)
and the third angle BAC to the third angle EDF.

Wherefore, if two triangles, &c. Q. E. D.

66. On Prop. XXVI.

This proposition is double. It contains two of the cases of equality of triangles; the first of which may be separately enunciated thus:

If two triangles have two angles of the one equal to two angles of the other, each to each, and also the sides adjacent to these angles equal; they shall have all their other elements equal in pairs.

The enunciation of the second case of Prop. XXVI. after this manner will be obvious.

These two cases may each, like Props. IV. and VIII., be proved by superposition, as shown in the Appendix.

Exercise XXX.

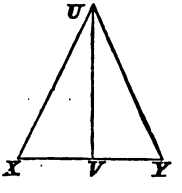
1. In the triangles PQR, XYZ, given angle P equal to X, and Q equal to Y, and base QR equal to YZ; which part of Prop. XXVI. applies to this case? (Figure on page 73.)

Give the remaining pairs of elements which are equal by that proposition.

2. Given, in the same triangles, that angles P and Q are respectively equal to Y and Z; which pair of sides must be equal in order that the first part of XXVI. may be applicable?

Write out the resulting pairs of equals also.

Questions 3—8 refer to the annexed diagram, in which it is given that UX, UY are equal.



3. If UV bisects the angle XUY, find two angles and a side in the triangle UVX which are respectively equal to two angles and a side of the triangle UVY. Let the side in each triangle be adjacent to the angles.

4. Also find two angles and a side in one triangle equal to similar parts in the other, but the sides being opposite a pair of equal angles.

5. Deduce from either case that UV bisects the base, and is perpendicular to it.

6. Next, given that UV is perpendicular to XY, find two angles and a side in one of the triangles respectively equal to two angles and a side in the other, so as to fulfil the conditions of the second part of XXVI.

7. Also find similar data, but taking a different pair of sides.

8. Infer from either set of data that the perpendicular bisects the base, and also the angle XUY.

9. If any angle be bisected by a straight line, any point in this bisector is equally distant from the two straight lines which contain the angle.

(By the distance of a point from a line is meant the perpendicular from the point to the line.)

10. Prove the first part of Prop. XXVI. with the trial hypothesis taken thus: "Let, if possible, AC be greater than DF."

67. Further Remarks on the Cases of Equality of Two Triangles.

The propositions which have now been studied in five sections constitute one of the greater divisions of the First Book of the Elements. Its principal subject is that of Triangles. The most important propositions are, in addition to XIII., those which treat of cases of equality "in all respects" between two triangles, namely IV., VIII., and XXVI.

The various sets of conditions under which this kind of equality subsists all require that three 'parts' of the one triangle shall be equal to three *similar* 'parts' of the other, each to each. Taking Euclid's order of the cases, these parts must be, in each triangle—

1. Two sides and the angle beween them—IV.
2. Two sides and the base (or the three sides)—VIII.
3. Two angles and the side adjacent to both—XXVI., 1st part.
4. Two angles and the side opposite one of them—XXVI., 2nd part.

If the learner were to try to make out other sets of three parts of a triangle, distinct from the above, he would find that there are two more; which it will be best to notice at once.

5. Two sides and an angle opposite one of them. These are insufficient, requiring a further condition.
6. Three angles. These may be easily shown to be insufficient also; but still the case is important in its own place.

Euclid does not enter into the consideration of either of these cases before Book VI.; but we shall bestow some little notice on each of them in future *exercises*.

Exercise XXXI.

1. Construct a triangle, given two sides and the angle which they contain.
2. Construct a triangle, given two angles and the side adjacent to both.
3. Construct an isosceles triangle, given base and altitude.
4. Construct an angle equal to three-fourths of a right angle.
5. If the perpendiculars drawn from the vertices of a triangle upon the opposite sides are all equal, show that the triangle is equilateral.

Examination XXII.

1. What propositions form the first of the greater divisions of Book I.? Give the general subject of this division, and name its four most important propositions. 2. Give one enunciation which shall include those of

V. and XVIII. 3. Give an enunciation which shall include VI. and XIX. 4. Show in what respect the order of Props. XVIII. and XIX. resembles that of V. and VI. 5. When a foot-passenger crosses a street slant-wise, of which proposition of Euclid does he show a *practical* knowledge? 6. Enunciate a proposition which shall include IV. (first result) and XXIV. 7. Enunciate a proposition which shall include VIII. and XXV. 8. Find a particular case of XXIII. which is previously given. 9. Construct according to XXII., when the *data* are three straight lines respectively 3 inches, 2 in., and 1 in. long; and state the result. 10. In the enunciation of the first part of Prop. XXVI. state the hypothesis. 11. Enunciate the second part of the proposition, and state its hypothesis and thesis separately. 12. Show that the thesis in each case, as given in the text, contains three separate conclusions. What other conclusion might have been added with the same proof? 13. State the nature of each proof. Give trial hypothesis used in each part.

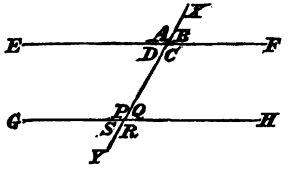
SECTION VI.—PROPOSITIONS XXVII.—XXXI.

Parallelism.

68. ANGLES MADE BY A STRAIGHT LINE CUTTING TWO OTHERS WHICH DO NOT THEMSELVES INTERSECT.

Suppose XY a straight line cutting two others EF, GH, but not forming a triangle with them.

At each point of intersection are four angles. Denote the first set of four by A, B, C, D respectively; the second, by P, Q, R, S.



We shall, for the future, speak of

A, B, R, S as *exterior* angles;

C, D, P, Q as *interior* angles;

C, P as *alternate* angles, and similarly D, Q;

and P will be called the *opposite* interior angle to A *on the same side of the line* XY; similarly Q to B, &c.

Exercise XXXII.

Questions 1—14 refer to the figure of this article.

1. Divide the eight angles into two divisions, separated from each other by the line XY.

2. Name the interior angles which are on the same side of XY as A; also as B.

3. What interior angle is adjacent to A?—to R?

4. State the angle opposite to R on the same side of the intersecting line; also to S.

5. If D, Q are equal, show that B is equal to Q.

6. If A, P are equal, show that C and its alternate angle are also equal.

7. What proposition shows that C, D are together equal to two right angles?

8. If C, Q are together equal to two right angles, show that D and its alternate angle are equal.

9. If C is equal to its alternate angle, show that A is equal to its opposite interior angle on the same side of the intersecting line.

10. If R is equal to its opposite interior angle on the same side, show that Q is equal to its alternate angle.

11. If A, P are equal, show that B, C, D are respectively equal to Q, R, S.

12. If A, P are equal, show that either pair of interior angles which are on one side of XY are together equal to two right angles.

13. If angles D, P are together less than two right angles, show that FE, HG, when produced ever so far, will meet towards E, G. (Apply Axiom 12.)

14. If angles C, Q are together less than two right angles, show that EF, GH are not parallel. In what direction will they meet?

15. In the figure to Prop. XXVIII., what angle is alternate with BGH? Which is the opposite interior angle to CHF? Which are the two interior angles on the same side as A?

16. In the second figure to XXVII., which is the greater angle, EFC or GEF? Give the reference.

17. In the first figure of XXVII., if AEF, EFD are equal, will AB meet FD when produced to the right? If not, say why.

69. Proposition XXVII.—Theorem.

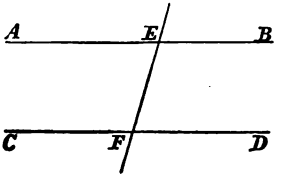
If a straight line, falling on two other straight lines, makes the alternate angles equal to each other; these two straight lines shall be parallel.

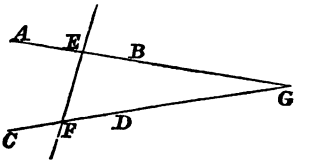
Let the straight line EF,
which falls upon the two straight lines AB, CD,
make the alternate angles AEF, EFD equal to one another.

Then AB shall be parallel to CD.

For, if AB be not parallel to CD,
AB and CD being produced will meet
either towards A and C, or towards B and D.

If possible, let AB, CD produced meet towards B and D in G.





Then GEF is a triangle,
and its exterior angle AEF is greater than the opposite interior angle EFG. (I. 16)
But the angle AEF is equal to the angle EFG; (Hyp.)
hence AEF is both greater than and equal to EFG,
which is impossible.

Therefore AB, CD produced do not meet towards B, D.
In like manner it may be demonstrated,
that they do not meet when produced towards A, C.
Therefore AB is parallel to CD. (Def. 35)
Wherefore, if a straight line, &c.

Q. E. D.

70. Proposition XXVIII.—Theorem.

If a straight line falling upon two other straight lines, either makes the exterior angle equal to the interior and opposite upon

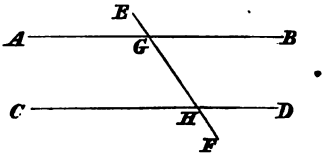
the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line *EF*, which falls upon the two straight lines *AB*, *CD*,

make the exterior angle *EGB* equal to the interior opposite angle *GHD* upon the same side;

or make the two interior angles *BGH*, *GHD* on the same side together equal to two right angles.

Then, in either case, AB shall be parallel to CD.



In the first case,

because the angle *EGB* is equal to the angle *GHD*, (Hyp.)

and *EGB* is also equal to the angle *AGH*; (I. 15)

therefore AGH is equal to GHD; (Ax. 1)

and these are alternate angles.

Therefore *AB* is parallel to *CD*. (I. 27)

In the second case,

because the angles *BGH*, *GHD* are together equal to two right angles, (Hyp.)

and that the angles *AGH*, *BGH* are also together equal to two right angles; (I. 13)

therefore *AGH* and *BGH* are equal to *BGH* and *GHD*. (Ax. 1)

Take away the common angle *BGH*;

then the remaining angle AGH is equal to the remaining angle GHD; (Ax. 3)

and these are alternate angles.

Therefore *AB* is parallel to *CD*. (I. 27)

Wherefore, if a straight line, &c.

Q. E. D.

Exercise XXXIII.

1. Show that XXVIII. is a double proposition, and enunciate the two separate propositions of which it is made up.

2. Enunciate a proposition which shall have a triple hypothesis, and include both XXVII. and XXVIII.

3. Write a converse to the proposition just referred to which shall therefore have a triple conclusion.

4. If three straight lines placed so as to form the letter Z contain two equal angles, show that two of them are parallel.

5. Show that two straight lines drawn perpendicular to the same straight line are parallel.

6. If the angles of a quadrilateral be all right, show that it is a parallelogram.

71. Proposition XXIX.—Theorem.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side, and likewise the two interior angles upon the same side together equal to two right angles.

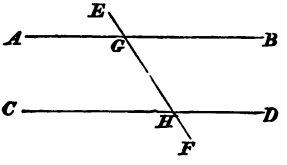
Let the straight line EF fall upon the parallel straight lines AB, CD, intersecting them in G and H.

Then the alternate angles AGH, GHD shall be equal,

the exterior angle EGB shall be equal to the opposite interior angle GHD upon the same side of EF,

and the two interior angles BGH, GHD, upon the same side of EF, shall be together equal to two right angles.

For, if the angle AGH be *not* equal to the angle GHD, one is greater than the other; let AGH be the greater;



and add to each of these unequals the angle BGH;

then AGH and BGH are greater than BGH and GHD. (Ax. 4)

But AGH and BGH are equal to two right angles; (I. 13)
therefore BGH and GHD are less than two right angles;
but those straight lines which, with another falling upon them, make the interior angles on the same side of it together less than two right angles, will meet if continually produced; (Ax. 12)
therefore AB, CD, if produced far enough, will meet.
But they never meet, since they are parallel: (Hyp.)
therefore the angle AGH is not unequal to GHD;
that is, AGH is equal to *its alternate angle* GHD.

But the angle AGH is equal to the angle EGB; (I. 15)
therefore, also, EGB is equal to its *opposite interior angle* GHD.
Add to each of these the angle BGH;
then EGB and BGH are equal to BGH and GHD; (Ax. 2)
but EGB and BGH are equal to two right angles; (I. 13)
therefore also *the two interior angles* BGH, GHD are together equal to two right angles.

Wherefore, if a straight line, &c.

Q. E. D.

Exercise XXXIV.

(In 2—6, Diagram II. is referred to; in 7—9, Diagram III.)

1. Between angles belonging to the figure of Art. 68 deduce as many equalities as you can by the application of (*a*) first part of XXIX., (*b*) second part of the same, and (*c*) third part.

2. If the angles EFG, EHG are equal, and FH bisects each of them, show, by Prop. XXVII., that EF is parallel to GH, and FG to EH.

3. If ABC, ADC are equal angles, and BD bisects each of them, prove $ABCD$ a parallelogram.

4. The angles BRL, RSM are equal; show that RL, SM are parallel.

5. If PY, QX, BD be all parallel, show that the triangles APY, AQX, ABD are equiangular to one another.

6. If $EFGH$ be a parallelogram, show that the angles adjacent to FH in the two triangles EFH, GFH form two pairs of equal angles. By further application of XXVI., show that those triangles are equiangular and have opposite sides equal.

7. The angles at A and B are right; apply XXVIII. to show that AD is parallel to BC.

8. If EF, GH be parallel to PQ, show that the angles EFG and HGU are each equal to PQF. Deduce that EF, GH are parallel.

9. If $EFUV$, $HGUV$ are parallelograms, show that $EFGH$ is a parallelogram.

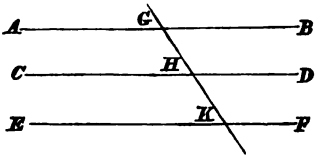
10. Let BC be a straight line; at B and C make equal angles CBA, BCD, on opposite sides of BC; then show that AB and CD are parallel.

72. Proposition XXX.—Theorem.

Straight lines which are parallel to the same straight line are parallel to one another.

Let the straight lines AB, CD be each of them parallel to EF.

Then shall AB be also parallel to CD.



Let the straight line GHK cut AB, CD, EF.

Then, because GHK cuts the parallel straight lines AB, EF, therefore the angle AGH is equal to the alternate angle HKF; (I. 29)

also because GHK cuts the parallels CD, EF,

therefore the exterior angle GHD is equal to the interior angle HKF. (I. 29)

And it was shown that AGH is equal to HKF;

therefore the angle AGH is equal to the angle GHD;

and these are alternate angles.

Therefore AB is parallel to CD. (I. 27)

Wherefore, straight lines which are, &c.

Q. E. D.

Note.—The case in which EF lies between AB and CD needs no demonstration; otherwise, within an angle contained by AB and CD, it would be possible for EF to lie without cutting, when produced without limit, either AB or CD.

73. Proposition XXXI.—Problem.

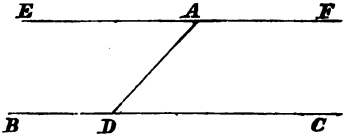
To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, and BC the given straight line.

It is required to draw a straight line through the point A parallel to the straight line BC.

In the line BC take any point D, and join AD;
at the point A, in the straight line AD,
make the angle DAE equal to the angle ADC; (I. 23)
and produce the straight line EA to F.

Then EF shall be parallel to BC.



Because AD meets the two straight lines EF, BC,
and makes the alternate angles EAD, ADC equal;
therefore EF is parallel to BC. (I. 27)

Wherefore, through the given point A has been drawn a straight line EAF parallel to the given straight line BC.

Q. E. F.

74. "Playfair's" Axiom of Parallelism.

It may be inferred, from previous observations on Axiom 12, that it is desirable to replace it by an axiom of a simpler nature, if that be possible. Many attempts have been made to accomplish this; one of which has met with so much approval in our own time that it is worth while to become acquainted with it.

The axiom proposed may be expressed thus:

Two straight lines which intersect cannot both be parallel to a third.

From this the usual Ax. 12 may very easily be deduced; and it will further clear up the nature of this important part of our subject to study some such proof as the following one:—

Given "Playfair's" Axiom, to prove Euclid's "Twelfth" Axiom.

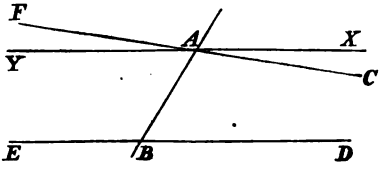
Let AC, BD be two straight lines, and AB another which falls upon them,

and let the two interior angles CAB, ABD on the same side of it be together less than two right angles.

Then AC, BD will meet, if produced, towards C, D.

Make the angle BAX equal to ABE; (I. 23)

so that AX is parallel to BD. (I. 27)



Then DBA, BAX are equal to DBA, ABE;

and therefore to two right angles. (I. 13)

But DBA, BAC are less than two right angles;

therefore BAX is greater than BAC;

that is, AC falls within the angle BAX,

and therefore AF falls beyond XY from B.

Now XY, FC intersect in A,

and XY is parallel to ED; (Constr.)

therefore FC meets ED. (Playfair's Axiom)

But AF has been proved to fall beyond XY,

therefore *FC meets BD on the same side of AB as C.* Q. E. D.

It has been objected to Axiom 12 that its converse is proved as Prop. XVII. It is similarly true of Playfair's Axiom that its converse is a proposition of Euclid, namely XXX.

Exercise XXXV.

1. How many distinct propositions are contained in XXVII., XXVIII., and XXIX.? What relation subsists among them?

2. Props. XXVII., XXVIII., and XXX. may be said to establish four tests of the parallelism of two given straight lines. State these tests.

3. What proposition may be stated thus: "Two straight lines which are both parallel to a third cannot intersect"?

State the relation of this proposition to "Playfair's" Axiom.

4. Point out a mode of solving XXXI. by successive applications of XII. and XI., supposing the given straight line unlimited in length.

5. Prove XXVIII., denoting the angles by letters, as in the figure of Art. 68.

6. Prove XXIX. similarly. 7. Similarly prove XXX.

8. Indicate a proof of XXVII. by means of XVII. applied in place of XVI.

9. Also of XXX. by the application of the first part of XXVIII. in place of XXVII.

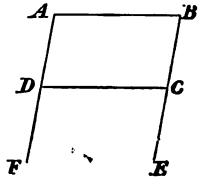
10. Also of XXX. by the application of the second and third parts of XXIX. and of the second part of XXVIII.

11. Show how to solve XXXI., using the first part of XXVIII. in place of XXVII.

SECTION VII. PROPOSITION XXXII. AND COROLLARIES.

Interior and Exterior Angles of Rectilineal Figures.

EXERCISE XXXVI.



In 1—6 the parallelogram $ABCD$, having two sides produced to E and F, is referred to.

1. Show that the angles DCE, ABC are equal.

2. Also the angles ADC, DCE.

3. Infer that the angles B, ADC are equal, quoting an axiom.

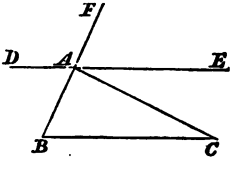
4. Prove that the angles A and BCD are equal.

5. Show that the angles A, B are together equal to two right angles. (Apply third part of XXIX.)

6. Prove that the angles BCD, CDA are together equal to two right angles.

7. Show that all the angles of a parallelogram are equal to four right angles.

The remaining questions refer to the triangle ABC, which has BA produced to F, and DAE drawn parallel to BC.



8. What is the sum of the angles DAB, BAC, CAE?

9. Show that the same angles are respectively equal to B, BAC, and C of the triangle.

10. Infer that the angles of the triangle ABC are together equal to two right angles.

11. Which angle of the figure is an exterior angle of the triangle?

12. State the two angles of which CAF is the sum? Which of these is equal to B, and which to C?

13. Show that the whole angle CAF is equal to B and C together.

14. Add CAB to each of these equals, and from the result draw an inference with regard to the sum of the angles of the triangle ABC.

75. Proposition XXXII.—Theorem.

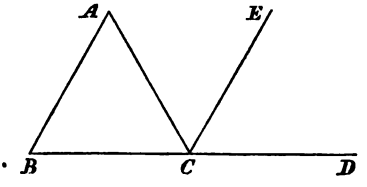
If a side of a triangle be produced, the exterior angle is equal to the two opposite interior angles; and the interior angles of every triangle are together equal to two right angles.

Let ABC be a triangle, having one side BC produced to D.

Then the exterior angle ACD shall be equal to the two opposite interior angles CAB, ABC;

and the three interior angles ABC, BCA, CAB shall be equal to two right angles.

Through C draw CE parallel to the side BA. (I. 31)



Then, because AC meets the parallels CE, BA,
the angle ACE is equal to the alternate angle BAC. (I. 29)
Also, because BC falls upon the parallels CE, BA,
therefore the exterior angle ECD is equal to the opposite interior angle ABC. (I. 29)
But the angle ACE was shown to be equal to BAC;
therefore the whole exterior angle ACD is equal to the two opposite interior angles CAB, ABC. (Ax. 2)
To each of these equals add the angle ACB;
then the angles ACD, ACB are equal to the three angles CAB, ABC, and ACB; (Ax. 2)
but ACD, ACB are equal to two right angles; (I. 13)

therefore, also, the angles CAB, ABC, and ACB are equal to two right angles. (Ax. 1)

Wherefore, if a side of a triangle be produced, &c.

Q. E. D.

Exercise XXXVII.

1. If a triangle be equilateral, determine the magnitude of each angle in terms of a right angle.

2. If a triangle be both right-angled and isosceles, determine the value of each acute angle.

3. If a right-angled triangle have one of its acute angles one-third of a right angle, determine the other one.

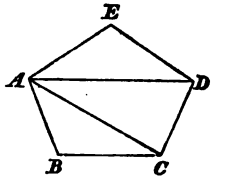
4. If, in the triangles *ABC*, *DEF*, the angles *A*, *D* are equal, and also *B*, *E*; show that *C* is equal to *F*.

5. If an exterior angle of a triangle be three-fourths of a right angle, and one of the opposite interior angles is one-fourth of a right angle; determine the other opposite interior angle.

6. In the figure to Corollary 1 (next page), show that all the angles of the triangles *ABF*, *FBC* are together equal to the angles of the quadrilateral *ABCF*.

7. Show that the sum of the angles of *ABCF* is equal to four right angles.

8. The given five-sided figure is divided into three triangles; how many right angles are there in the sum of the angles of those three triangles?



9. Show that the polygon *ABCDE* has all its angles together equal to all the angles of the three triangles *ABC*, *ACD*, *ADE*.

10. Show that the sum of the angles of the polygon *ABCDE* is equal to six right angles.

11. Make a six-sided figure *ABCDEF*, and divide it into triangles by drawing straight lines from one angle to each of the others. How many triangles are thus formed; and what is the sum of all their angles?

12. Show that the sum of the angles of a six-sided figure is equal to eight right angles.

13. Divide a five-sided figure into triangles by drawing straight lines from each angle to a point within the figure, as in the figure of Corollary 1, which follows. How many triangles do you thus obtain, and what is the sum of all their angles?

14. Say what is the sum of the angles around the vertex *F*, giving reference.

15. Considering F the common vertex of all the triangles, deduce from 13 and 14 above the sum of all the angles at the bases of those triangles.

16. Compare the sum of the angles of the polygon, and the sum of the base-angles referred to in 15. Infer the sum of the angles of the polygon in right angles.

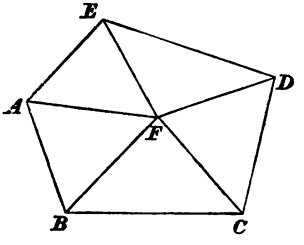
76. Simson's Corollaries to Proposition XXXII.

The Corollaries which follow might fairly be styled 'propositions,' having regard either to their importance or their degree of difficulty. The difficulty which the learner may experience is, however, like that of the Fourth Proposition, rather a matter of words and phraseology than of ideas. A special effort should therefore be made to gradually acquire command of the words; taking care, of course, to follow their meaning at the same time; and the difficulty will then be much lessened.

77. Corollary 1.—Theorem.

All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

For any rectilineal figure $ABCDE$ can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within it to each angle.



Then, because the three angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides,

therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides.

But the interior angles of the figure, together with the angles at the point F, are equal to all the angles of these triangles;

and the angles at F are equal to four right angles;

(I. 15, Cor. 2)

therefore the angles of the figure, together with four right angles, are equal to the angles of all the triangles.

But it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides.

Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides. Q. E. D.

Exercise XXXVIII.

1. If a figure be a pentagon, or five-sided, how many right angles make twice as many as the figure has sides? What must be added to the sum of the angles of the pentagon in order to make it equal to this number of right angles?

2. How many right angles are therefore equal to the sum of the angles of a pentagon?

3. If a figure be a hexagon, or six-sided, what number of right angles are equal to its six angles and four right angles together?

4. Deduce the sum of the angles of a hexagon.

5. Similarly obtain the sum of the angles of any quadrilateral.

6. Show that the second part of Prop. XXXII. is a particular case of Corollary I.

7. If a quadrilateral have three angles right, what will be the magnitude of the fourth?

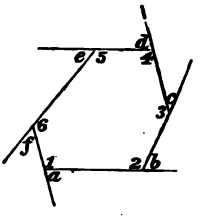
8. What proposition is equivalent to this rule?—

> To find the sum of the angles of any rectilineal figure, take twice as many right angles as the figure has sides, and subtract four right angles.

9. Apply this rule to find the sum of the angles of a hexagon. From the result deduce the magnitude of each angle when the hexagon is equiangular.

10. Similarly find the magnitude of each angle of an equiangular pentagon, octagon, decagon, duodecagon, and quindecagon respectively; of which the last four have eight, ten, twelve, and fifteen sides respectively.

11. Let there be a six-sided figure as shown, having all its sides produced in one direction; and denote the exterior angles thus formed by a, b, c, d, e, f, and the interior angles by 1, 2, 3, 4, 5, 6. What is the value of each of the pairs of angles: a and 1, b and 2, c and 3, &c. ?



12. What is the sum of the six interior and the six exterior angles taken altogether?

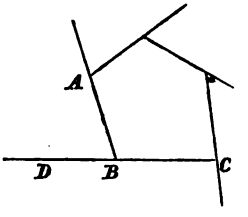
13. Assuming the result of Question 4 of this exercise, deduce from 12 the sum of the outside angles a, b, c, d, e, f taken alone.

14. After the manner indicated in Questions 11—13, calculate the sum of the exterior angles of a five-sided figure, such as that of Corollary 2, which follows.

15. Apply a similar process of reasoning to obtain the sum of the exterior angles of a decagon formed by producing its sides successively in *one* direction.

78. Corollary 2.—Theorem.

All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.



Since every interior angle ABC, and its adjacent exterior angle ABD, are together equal to two right angles, (I. 13)

therefore all the interior angles, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides.

But, by the foregoing Corollary, all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides;

therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles. (Ax. 1)

Take from these equals all the interior angles;

therefore all the exterior angles of the figure are equal to four right angles. (Ax. 3)

Q. E. D.

79. THE DETERMINATION OF THE MAGNITUDES OF THE ANGLES OF EQUIANGULAR POLYGONS.

In the preceding exercise we have practised calculations of the angles of equiangular polygons by means of Corollary 1. The procedure we followed renders clear to the mind the nature of that Corollary; and it also appears to be that which is usually preferred for the purpose to which it was applied. It will probably be found, however, that Corollary 2 affords a readier means of effecting the same calculations. This is certainly the case in such as require the angles to be expressed in degrees. The student may judge for himself while working the ensuing exercise.

EXERCISE XXXIX.

Note.—In this exercise the exterior angles of a figure are supposed to be made by producing its sides successively in the *same direction*.

1. If the interior angles of a figure be all equal, prove that its exterior angles are also equal.

2. If an exterior angle be one-third of a right angle, find the *adjacent* interior angle.

3. If an exterior angle of an equiangular figure be $\frac{4}{5}$ right angle, calculate each interior angle.

4. Calculate each exterior angle of an equiangular pentagon. Thence deduce each interior angle.

5. By the procedure indicated in 4, calculate each interior angle of an equiangular hexagon.

6. Prove this rule:—

> To find the interior angle of an equiangular polygon, divide four right angles by the number of sides, and subtract the result from two right angles.

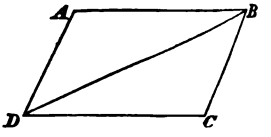
7. Apply this rule in finding the angle of an equiangular octagon, decagon, duodecagon.

8. Find the same angles in the same way, calling four right angles 360°, and obtaining each result in degrees.

SECTION VIII. PROPOSITIONS XXXIII., XXXIV.

Parts of a Parallelogram.

EXERCISE XL.



1. Suppose $ABCD$ a quadrilateral, having AB, CD parallel, and divided into two triangles by means of the diagonal BD. Find a pair of angles which are equal, one being in each triangle.

2. Given that AB, CD in the same figure are also equal, find two sides in the one triangle respectively equal to two sides in the other.

3. Show, by I. 4, that these triangles are equal in all respects.

4. Write out all the resulting equalities established by I. 4; and state which of them proves that AD, BC are parallel.

5. Which result proves that the same sides are also equal?

6. Prove that the opposite sides of a rhombus are parallel.

Note.—If two parallel straight lines be placed so as to be horizontal, those ends which are both to the right would be said to be "towards the same parts," and similarly those which are on the left. The expression would be applied to other positions of the lines in a similar manner.

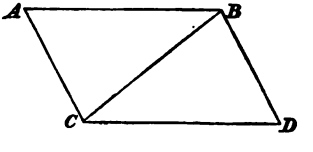
80. PROPOSITION XXXIII.—THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD.

Then AC, BD shall be equal and parallel.

Join BC.



Then, because AB is parallel to CD, and BC meets them, the angle ABC is equal to the alternate angle BCD. (I. 29)

And because AB is equal to CD, and BC common to the triangles ABC, DCB;

the sides AB, BC are equal to DC, CB, each to each;

and the angle ABC was proved equal to DCB;

therefore the base AC is equal to the base BD. (I. 4)

Also the angles are equal to which the equal sides AB, CD are opposite;

therefore the angle ACB is equal to CBD. (I. 4)

And because BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal;

therefore AC is parallel to BD. (I. 27)

And AC was shown to be equal to BD.

Therefore, the straight lines which, &c.

Q. E. D.

Exercise XLI.

1. What proposition may be, in part, enunciated thus?—

Any quadrilateral having two sides equal and parallel is a parallelogram.

How much more than this is proved in the proposition referred to?

2. Show that, if two opposite sides of a quadrilateral are equal and parallel, the diagonal will bisect it.

3. In the figure to XXXIII., given AB, CD parallel, and also AC, BD; find two angles in the triangle ABC equal to two in BCD, each to each.

4. Show that the side adjacent to these angles of the one triangle is equal to the side adjacent to those of the other.

5. Draw three inferences by I. 26.

6. From the two equalities of Question 3, deduce that the angles ABD, ACD are equal, quoting an axiom.

7. If on opposite sides of a base two equilateral triangles be made, the two together form a parallelogram.

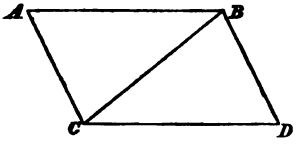
8. If two opposite sides of a quadrilateral be equal and parallel, the straight line bisecting both will be parallel to the remaining sides of the figure.

81. Proposition XXXIV.—Theorem.

The opposite sides and angles of a parallelogram are equal to one another; and either diameter bisects it, that is, divides it into two equal triangles.

Let $ABDC$ be a parallelogram, of which BC is a diameter.

Then its opposite sides and angles shall be equal, and its diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them,
the angle ABC is equal to the alternate angle BCD. (I. 29)
And because AC is parallel to BD, and BC meets them,
the angle ACB is equal to the alternate angle CBD. (I. 29)

Hence the triangles ABC, CBD have
the two angles ABC, BCA in the one
equal to the two BCD, CBD in the other, each to each;
and one side BC, adjacent to these equal angles, common;
therefore their other sides are equal, each to each, and also their third angles; (I. 26)
namely, AB equal to CD, AC to BD, and the angle BAC to the angle BDC;

and these are the opposite sides and one pair of opposite angles of the parallelogram.

Also the angle ABC has been proved equal to BCD,
and the angle CBD to ACB;
therefore the whole ABD is equal to the whole ACD; (Ax. 2)

and these are the other opposite angles of the figure.

Hence the opposite sides and angles of a parallelogram are equal to one another.

Again, in the triangles ABC, BCD,
since AB is equal to CD and BC common,

the sides AB, BC are equal to DC, CB, each to each;
and the angle ABC was proved to be equal to DCB;
therefore the triangles ABC, BCD are equal. (I. 4)
That is, the diameter BC bisects the parallelogram $ACDB$.
Wherefore, the opposite sides, &c.

Q. E. D.

Exercise XLII.

1. Show how we might superpose one of the triangles in the figure of XXXIV. upon the other.
2. Show how an extended enunciation of Prop. XXVI. would enable us to shorten the demonstration of XXXIV.
3. One angle of a parallelogram is half a right angle. Find the others.
4. One angle of a parallelogram is one-third of two right angles. Find the remaining angles.
5. If the parallelogram just referred to be equilateral, prove that one of its diagonals will divide it into two equilateral triangles.
6. If one angle of an equilateral quadrilateral be right, show that the figure is a square.

Exercise XLIII.

(Questions 2—6 refer to Diagram II., 7—9 to Diagram IV.)

1. State a fourth condition, derived from XXXIII., upon which two straight lines will be parallel.
2. PQ, UV are equal and parallel; name two other lines which must also be equal and parallel.
3. Given $ABCD$ a parallelogram, write out the consequences arising by XXXIV.
4. Write out similar consequences with regard to the parallelogram $EHDX$.
5. Given $APEY$, $PQEY$, both parallelograms; prove AQ bisected in P.
6. Also prove that the triangles APY, PQE are equal.
7. AB is equal to AO and parallel to OC; prove that BC is equal and parallel to OA.
8. AF, in the same, is also equal to OA and parallel to OE; prove $AFEO$ a parallelogram.
9. Given the results of 7 and 8 foregoing, prove BF equal and parallel to CE.

SECTION IX.—PROPOSITIONS XXXV.–XLII.

Parallelism in relation to Equivalent Parallelograms and Triangles.

82. The Peculiarities of Proposition XXXV.

The manner in which the next proposition is proved deserves notice on three points.

Firstly, we have Axioms 2 and 3 used in a modified form, which we may express thus (see Art. 19 also):—

2. If the same thing be added to equals, the wholes are equal.

3. If the same be taken from equals, the remainders are equal.

Secondly, we have Axiom 3 further modified so as to be equivalent to the following:—

> If two different parts of one whole be equal, and if one be taken away and restored again, and *then* the other be taken away, the remainder after the first operation will be equal to the remainder after the second.

Thirdly, the principal demonstration is made to apply, excepting in one line, to two figures at the same time; yet not altogether in the same way. This occurs here for the first time with us; it often occurs in Geometry, but will not do so again in this book.

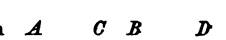
The two figures above referred to should be drawn by the learner together; and it should be carefully observed how each step of the argument applies to both.

Exercise XLIV.

1. Given AB, CD equal; enunciate an axiom which proves AC equal to BD; and state in which of Euclid's axioms it is included.



2. Given AB and CD equal again; enunciate an axiom proving AC equal to BD. Which of Euclid's axioms includes it?

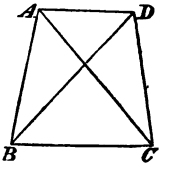


3. If we say, in reference to the lines above given, "The whole, or the remainder, AC is equal to the whole, or the remainder, BD"; which of these words will refer to the upper line alone, which to the second alone, and which to both?

4. In the annexed figure, from the trapezium $ABCD$ take the triangle ACD; what is left?

5. Having put ACD back again, take ABD away; what is left?

6. Given, in the same figure, that the triangles ABC, DBC are equal, apply a form of Axiom 3 to prove that the triangles ABD and ACD are also equal.



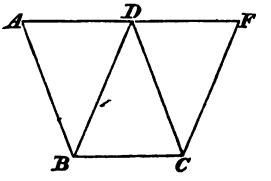
Note.—That part of Proposition XXXV. which *precedes* the first figure should be read with reference to all the figures. If the second part of the proposition were taken first, the order would be improved in respect of convenience.

83. Proposition XXXV.—Theorem.

Parallelograms upon the same base and between the same parallels are equal to one another.

Let the parallelograms $ABCD$, $EBCF$ be upon the same base BC, and between the same parallels AF, BC.

Then these parallelograms $ABCD$, $EBCF$ shall be equal.

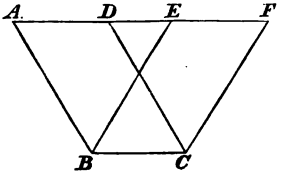


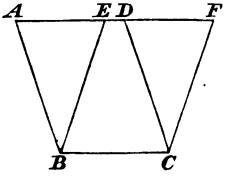
First, let the sides which are opposite the base BC be terminated in the same point,

so that E coincides with D, and the parallelograms become $ABCD$, $DBCF$.

Then it is plain that each parallelogram is double of the triangle BDC; (I. 34)

therefore $ABCD$ is equal to $DBCF$. (Ax. 6)





Secondly, let the sides AD, EF, opposite the base BC, be not terminated in the same point.

Then, because $ABCD$ is a parallelogram,

therefore AD is equal to BC; (I. 34)

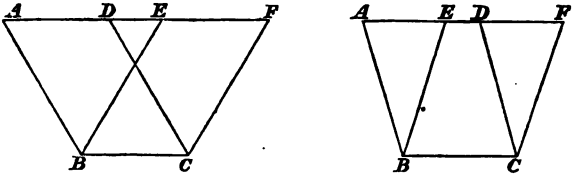
and, for a similar reason, EF is equal to BC;

wherefore AD is equal to EF. (Ax. 1)

Therefore the whole, or the remainder, AE is equal to the whole, or the remainder, DF. (Ax. 2, or 3)

Also, *AB* is equal to *DC*; (I. 34)

hence, in the triangles, *FDC*, *EAB*,



FD, *DC* are equal to *EA*, *AB*, each to each;

and the exterior angle *FDC* is equal to the opposite interior angle *EAB*; (I. 29)

therefore the triangle FDC is equal to the triangle EAB. (I. 4)

From the trapezium *ABCF* take the triangle *FDC*; the parallelogram *ABCD* remains:

from the *same* trapezium take the triangle *EAB**; the parallelogram *EBCF* remains;

and these remainders are equal; (Ax. 3)

that is, the parallelograms *ABCD*, *EBCF* are equal.

Therefore, parallelograms upon the same base, &c.

Q. E. D.

EXERCISE XLV.

1. By means of I. 29, find two angles in *FDC* which are equal to two in *EAB*, each to each.

What proposition shows that the remaining angles in those triangles are equal?

2. Given that these remaining angles are equal, prove the triangles *FDC*, *EAB* are equal without using Axioms 2 or 3. (Apply I. 4 to the triangles.)

3. In the figure of XXXVI., if *EB* be equal and parallel to *CH*, what must be true of *EH*, *BC* according to XXXIII.?

4. What would then follow with regard to the figure *EBCH*?

5. If *EBCH* and *EFGH* be said to be upon the same base, which line will be meant?

6. Prove these figures equal if they are parallelograms and *BG* is one straight line.

* For this operation the triangle *FDC* must be supposed to have been put back again.

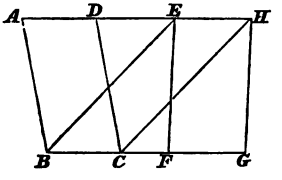
84. PROPOSITION XXXVI.—THEOREM.

Parallelograms upon equal bases and between the same parallels are equal to one another.

Let *ABCD*, *EFGH* be parallelograms upon equal bases *BC*, *FG*, and between the same parallels *AH*, *BG*.

Then ABCD shall be equal to EFGH.

Join *BE*, *CH*.



Because *BC* is equal to *FG* (Hyp.), and *FG* to *EH*; (I. 34)
therefore *BC* is equal to *EH*. (Ax. 1)
Also, these lines are parallels,
and joined towards the same parts by *BE*, *CH*.
But straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33)
therefore BE, CH are both equal and parallel;
wherefore *EBCH* is a parallelogram. (Def. A.)
Then, since the parallelograms *ABCD*, *EBCH* are upon the same base *BC*, and between the same parallels *BC*, *AH*;
therefore *ABCD* is equal to *EBCH*. (I. 35)
For a similar reason, *EFGH* is equal to *EBCH*.
Hence the parallelograms ABCD, EFGH are equal to one another. (Ax. 1)
Therefore parallelograms upon equal, &c.

Q. E. D.

EXERCISE XLVI.

1. Prove XXXVI., by joining *AF*, *DG* instead of *BE*, *CH*.
2. Make the figure of XXXVI., and join *EC*; then name two triangles which are each half of *EBCH*.

3. Afterwards join *DB*; and name the figure which is double of the triangle *DBC*.

4. Show that *DBC* and *EBC* are halves of equal figures, and draw an inference by Ax. 7.

5. Join *EG* in the same figure, and show that *EFG* and *EBC* are halves of equal things; also draw the inference.

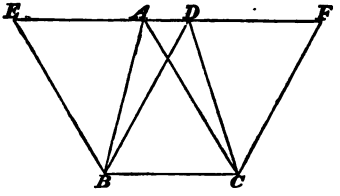
35. Proposition XXXVII.—Theorem.

Triangles upon the same base and between the same parallels are equal to one another.

Let the triangles *ABC*, *DBC* be upon the same base *BC*, and between the same parallels *AD*, *BC*.

Then ABC shall be equal to DBC.

Produce *AD* both ways to the points *E*, *F*;
through *B* draw *BE* parallel to *CA*, (I. 31)
and through *C* draw *CF* parallel to *BD*.



Then each of the figures *EBCA*, *DBCF* is a parallelogram;
and EBCA is *equal to DBCF*, (I. 35)
because they are upon the same base *BC*, and between the same parallels *BC*, *EF*.
And because the diameter *AB* bisects the parallelogram *EBCA*,
therefore the triangle ABC is half of EBCA. (I. 34)
Also, since the diameter *DC* bisects the parallelogram *DBCF*,
therefore the triangle DBC is half of DBCF;
but the halves of equal things are equal; (Ax. 7)
therefore the triangle *ABC* is equal to the triangle *DBC*.

Wherefore, triangles, &c.

Q. E. D.

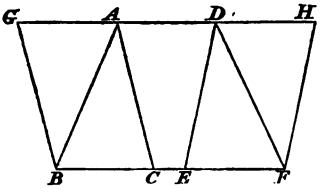
86. PROPOSITION XXXVIII.—THEOREM.

Triangles upon equal bases and between the same parallels are equal to one another.

Let the triangles *ABC*, *DEF* be upon the equal bases *BC*, *EF*, and between the same parallels *BF*, *AD*.

Then ABC shall be equal to DEF.

Produce *AD* both ways to *G* and *H*;
through *B* draw *BG* parallel to *CA*, (I. 31)
and through *F* draw *FH* parallel to *ED*.



Then each of the figures *GBCA*, *DEFH* is a parallelogram;
and they are equal to one another, (I. 36)
being upon equal bases and between the same parallels.
And because *AB* bisects the parallelogram *GBCA*,
therefore the triangle ABC is half of GBCA; (I. 34)
and because *DF* bisects the parallelogram *DEFH*,
therefore the triangle DEF is half of DEFH;
but halves of equal things are equal; (Ax. 7)
therefore the triangle *ABC* is equal to the triangle *DEF*.

Wherefore, triangles upon equal bases, &c.

Q. E. D.

EXERCISE XLVII.

1. In the figure to XXXVII. prove the triangles *ABE*, *DCF* equal.
2. Also the triangles *BAD*, *CAD*.
3. In the figure of XXXVIII. let *D* coincide with *A*. Will the two triangles be between the same parallels?
4. If the vertex of a triangle be joined to the middle of the base, show that the triangle is bisected.

5. If the base of a triangle be trisected, how may you trisect the triangle?

6. Given that the diagonals of a parallelogram bisect one another, show, by XXXVIII., that they divide the figure into four equal triangles.

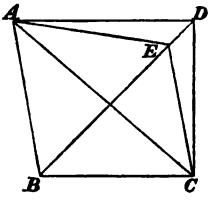
37. Proposition XXXIX.—Theorem.

Equal triangles upon the same base and upon the same side of it are between the same parallels.

Let the equal triangles *ABC*, *DBC* be upon the same base *BC*, and upon the same side of it.

Then ABC and DBC shall be between the same parallels.

Join *AD*.



AD shall be parallel to BC.

For, if it is not, through *A* draw *AE* parallel to *BC*, (I. 31)
meeting *BD*, or *BD* produced, in *E*,
and join *EC*.

Then the triangle ABC is equal to the triangle EBC, (I. 37)
because they are upon the same base *BC*,
and between the same parallels *BC*, *AE*.

But the triangle *ABC* is equal to the triangle *DBC*; (Hyp.)
therefore the triangle DBC is equal to the triangle EBC,
the greater equal to the less, which is impossible;
therefore *AE* is not parallel to *BC*.

In the same manner, it can be demonstrated,
that no other line but AD is parallel to BC.
AD is therefore parallel to *BC*.

Wherefore, equal triangles, &c.

Q. E. D.

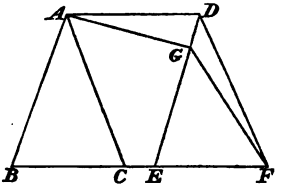
88. PROPOSITION XL.—THEOREM.

Equal triangles, upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the straight line BF,
and on the same side of BF.

Then they shall be between the same parallels.

Join AD.



AD shall be parallel to BF.

For, if it is not, through A draw AG parallel to BF, (I. 31)
meeting ED, or ED produced, in G,
and join GF.

Then the triangle ABC is equal to the triangle GEF, (I. 38)
because they are upon equal bases BC, EF,
and between the same parallels BF, AG.

But the triangle ABC is equal to the triangle DEF; (Hyp.)
therefore the triangle DEF is equal to the triangle GEF,
the greater equal to the less, which is impossible.

Therefore AG is not parallel to BF.

And in the same way it may be demonstrated,
that there is no other parallel to it but AD;
AD is therefore parallel to BF.

Wherefore equal triangles, &c.

Q. E. D.

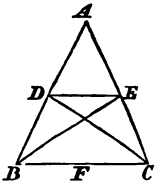
89. FURTHER TESTS OF PARALLELISM.

In addition to those we have observed in previous propositions, XXXIX. and XL. furnish most useful conditions of parallelism. It will be partly the aim of the next exercise to impress this view of them upon the memory.

Exercise XLVIII.

1. Prove XL. with a figure in which AG meets ED produced.
2. What proposition may be thus enunciated?—

 If two triangles which have a base common, and are both on the same side of it, be equal, their vertices will lie upon a line which is parallel to the base.
3. Give an enunciation to XL. similar to this one.
4. If there be three or more equal triangles upon one base and on one side of it, prove that their vertices will be in one straight line.
5. In the triangle ABC, let D, E be middle points of AB, AC. Show that the triangle BCD is half of ABC.



6. Similarly, show BEC is half of the same.
7. By what axiom are BDC, BEC equal? Draw an inference with regard to the lines DE, BC.
8. Take F the middle point of BC, and show that, if EF be joined, it is parallel to AB, and that, similarly, DF would be parallel to AC.

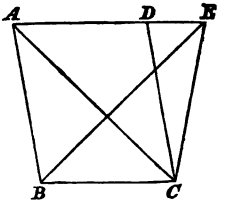
90. Proposition XLI.—Theorem.

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram is double of the triangle.

Let the parallelogram $ABCD$ and the triangle EBC be upon the same base BC and between the same parallels BC, AE.

Then the parallelogram $ABCD$ shall be double of the triangle EBC.

Join AC.



Then the triangle ABC is equal to EBC, (I. 37)
because they are upon the same base BC,
and between the same parallels BC, AE.

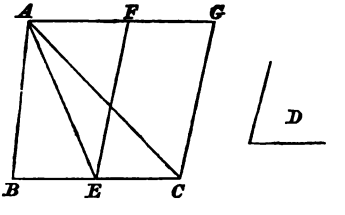
But the parallelogram ABCD is double of ABC,
because its diameter *AC* bisects it; (I. 34)
wherefore *ABCD* is also double of *EBC*.
Therefore, if a parallelogram, &c.

Q. E. D.

91. Proposition XLII.—Problem.

To describe a parallelogram that shall be equal to a given triangle and have an angle equal to a given rectilineal angle.

Let *ABC* be the given triangle,
and *D* the given rectilineal angle.
It is required to describe a parallelogram that shall be equal to ABC, and have an angle equal to D.
Bisect *BC* in *E* (I. 10), and join *AE*.
At the point *E*, in the straight line *EC*,
make the angle FEC equal to the angle D. (I. 23)
Through *A* draw *AFG* parallel to *BC*, (I. 31)
and through C draw *CG* parallel to *EF*.
Then the figure CEFG is a parallelogram. (Def. A.)



Because the triangles *ABE*, *AEC* are on equal bases *BE*, *EC*,
and between the same parallels *BC*, *AG*,
they are equal to one another; (I. 38)
therefore the triangle ABC is double of AEC.
But the parallelogram *FECG* is double of *AEC*; (I. 41)
because they are upon the same base *EC*,
and between the same parallels *EC*, *AG*.
therefore the parallelogram FECG is equal to ABC; (Ax. 6
and the angle *CEF* was made equal to *D*.

Wherefore, a parallelogram *FECG* has been described, equal to the given triangle *ABC*, and having one angle *CEF* equal to the given angle *D*.

Q. E. F.

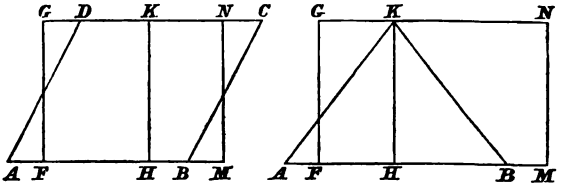
92. Altitudes of Triangles and Parallelograms.

As with the triangle, so with the parallelogram, any side may be chosen to be called the *base*. It will be convenient here, for practical applications of this part of the subject, to define *height or altitude* of a parallelogram and of a triangle.

Definition.—The altitude of a parallelogram is a perpendicular drawn to the base, or base produced, from a point in the opposite side, or the opposite side produced.

The altitude of a triangle is a similar perpendicular to the base, but is drawn from either the vertex, or from a point in a parallel to the base which passes through the vertex.

As examples, these figures *ABCD*, *ABK* are a parallelogram and a



triangle respectively, and *FG*, *HK*, *MN* are perpendiculars to *AB*; while, in the second figure, *GN* is parallel to *AB*. Then the altitude of either the parallelogram or the triangle may be taken to be any one of the three perpendiculars *FG*, *HK*, *MN*.

Wherever this altitude be thus taken, it will be found that it is always of the same magnitude. This, of course, is necessary, and its proof, being easy, will be required from the learner in the ensuing exercise.

Another explanation may conveniently be introduced here.

When two figures are equal in *area*, but not necessarily so in other respects, they are sometimes said to be "equivalent." Such figures have come under notice in the present section.

Exercise XLIX.

1. If from two points in one of a pair of parallels two perpendiculars be drawn to meet the other parallel, show that these perpendiculars shall be equal.

2. Show that the figure obtained, which is bounded by the parallels and the perpendiculars, is a rectangle.

3. Show that, if two parallelograms have the same altitude, they can be placed between the same parallels.

4. Hence show that parallelograms on equal bases and of equal altitudes are themselves equal.

5. Show that a triangle is half of the rectangle upon an equal base and of equal altitude.

6. Make a rectangle equal to a given triangle.

7. Show that a parallelogram is equal to a triangle between the same parallels, but on double its base.

8. Make a parallelogram equal to a given parallelogram, but having one angle at its base equal to half a right angle.

9. Make a rectangle equal to a given parallelogram.

10. Make a triangle equal in area to a given parallelogram.

11. Make a right-angled triangle equal to a given triangle.

12. Make a triangle equal to a given triangle, having an angle at its base equal to a given angle.

13. Show that the parallelograms of Prop. XXXV. are "equivalent" figures.

14. Of the following propositions, state which relate to figures "equal in all respects," and which to "equivalent" figures merely: XXXVIII., VIII., XL., IV., XXVI., XXXVI.

Note.—If convenient, it would be well to work the Simple Mensuration of the rectangle, parallelogram, and triangle at the same time with this part of Geometry.

93. SECTION X.—PROPOSITIONS XLIII.—XLV.

Complementary Parallelograms, and the Reduction of Rectilineal Figures to Parallelograms of equal area.

It is curious that in the next proposition two new terms are employed; and that definitions of both are incorporated with the proposition itself. These terms are—"parallelograms about the diameter of a parallelogram," and "complements."

A few general hints will here be given to introduce the latter term; and then the student will be left to extract the definitions which suit the proposition from itself.

If we take one or more parts of any whole magnitude, that *which must be added* to those parts to make up the whole, is sometimes called the "complement" *of the parts*.

For example, if we take two-thirds of a square, the remaining one-third might be called the complement of the *two-thirds*. But it must be specially noticed, that the one-third is *not* called the complement of the *whole* square.

To take an illustration in numbers, if we have under consideration at any time the number 10 as our whole magnitude, the complement of 3 would be 7; if the whole be unity, the complement of $\frac{1}{3}$ would be $\frac{2}{3}$; and so on.

EXERCISE L.

Throughout this exercise the figure of the next proposition is referred to, in which *ABCD*, *EH*, and *GF* are all parallelograms.

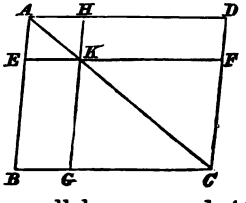
1. Taking *ABCD* as the whole magnitude, give the complement of the triangle *ABC*, of the parallelogram *BF*, and of the parallelogram *FG*.
2. What will be the complement of the figures *BK*, *KD* taken together?
3. Of the parallelograms *EH*, *GF* taken together?
4. Of the whole figure *ABCD*?
5. What proposition proves that the triangles *ABC*, *ADC* are equal? Also *AEK*, *AHK*?
6. Show that the triangles *AEK* and *KGC* are together equal to *AHK* and *KFC*.
7. Give the remainders when *AEK*, *KGC* are taken from *ABC*; also *AHK*, *KFC* from *ADC*.
8. By what axiom may these remainders be shown to be equal?

94. PROPOSITION XLIII.—THEOREM.

The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.

Let *ABCD* be a parallelogram, of which the diameter is *AC*;
and *EH*, *GF* the parallelograms about *AC*,
that is, which have *AC* passing through opposite angles in each.
Also let *BK*, *KD* be the other parallelograms which complete the whole figure *ABCD*,
and which are therefore called the complements.

Then the complements BK, KD shall be equal.



Because *ABCD* is a parallelogram, and *AC* its diameter;
therefore the triangle ABC is equal to ADC. (I. 34)

Again, because $AEKH$ is a parallelogram, and AK its diameter, therefore the triangle AEK is equal to AHK; (I. 34)

and, for the same reason, the triangle KGC is equal to KFC;

wherefore the two triangles AEK, KGC are equal to the two AHK, KFC. (Ax. 2)

But the whole triangle ABC is equal to the whole ADC;

therefore the remaining complement BK is equal to the remaining complement KD. (Ax. 3)

Wherefore, the complements, &c. Q. E. D.

95. EXERCISE LI.

1. Write the definition, as given in substance in XLIII., of "parallelograms about the diameter of a parallelogram."

2. Also of "complements" of those parallelograms.

3. Say why it would be wrong to describe BK, KD as the complements of $ABCD$ in the figure of XLIII.

4. Given a straight line AB and a triangle PQR, show how to make a triangle equal to PQR, one side of which shall be in AB produced, and one angle at B equal to a given angle.

5. Show how a parallelogram may now be made equal to PQR having one side in AB produced, and an angle at B equal to a given angle.

6. If one straight line meets two others without cutting them, how many angles are formed?

7. Under what condition will the two lines meet if produced, and in what direction must they be produced to meet?

8. In the figure to XLIV., FL, EM, GA are all parallelograms; show that, if BL be equal to C, then C is equal to BF.

9. Also show that, if the angle ABM is equal to the angle D, then D is equal to the angle EBG.

10. In the figure to XLIII., if the angle BAD be right, show that all the quadrilaterals of the figure are rectangles.

11. In the same figure, if EH be a square, so is the whole BD.

Definition.—A parallelogram is said to be applied to a straight line, when it is described upon it as one of its sides.

96. PROPOSITION XLIV.—PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line,

C the given triangle, and D the given rectilineal angle.

It is required to apply to AB a parallelogram equal to C, and having an angle equal to D.

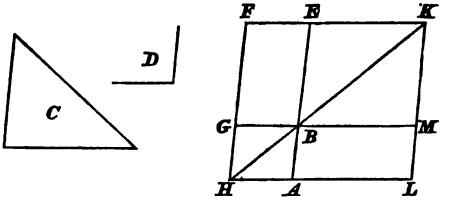
Make the parallelogram *BEFG* equal to *C*, and having the angle *EBG* equal to *D*, (I. 42)

so that *BE* may be in the same straight line with *AB*;

produce *FG* to *H*,

through *A* draw *AH* parallel to *EF*, (I. 31)

and join *HB*.



Then, because the straight line *HF* meets the parallels *AH*, *EF*,

therefore the angles *AHF*, *HFE* are together equal to two right angles; (I. 29)

wherefore the angles BHF, HFE are together less than two right angles;

therefore *HB*, *FE* will meet if produced. (Ax. 12)

Let them be produced and meet in *K*,

through *K* draw *KL* parallel to *FH*,

and produce *HA*, *GB* to meet *LK* in *L*, *M*.

Then BL shall be the parallelogram required.

Now *HLKF* is a parallelogram, (Constr.)

of which the diameter is *HK*;

also *AG*, *ME* are the parallelograms about *HK*,

and *LB*, *BF* are their complements;

therefore the parallelograms *LB*, *BF* are equal. (I. 43)

But *BF* is equal to the triangle *C*; (Constr.)

wherefore LB is equal to C.

Again, because the angle *EBG* is equal to *ABM*, (I. 15)

and likewise to the angle *D*; (Constr.)

therefore the angle ABM is also equal to D. (Ax. 1)

Wherefore to the straight line AB the parallelogram LB has been applied,
equal to the triangle C, and having the angle ABM equal to the angle D.

Q. E. F.

97. Exercise LII.

1. State the proposition which may be thus enunciated:
"Construct a parallelogram, given one side, one angle, and a triangle of equal area."

2. Show how to make a rectangle, having given one side, and also a triangle of equal area.

3. Prove any two adjacent angles of a parallelogram are together equal to two right angles. Give examples from the figure of XLV., in which FH, HL are each a parallelogram.

(Questions 4–9 refer to the same figure.)

4. Given also that the angles K and L are each equal to E, show that GHM is equal to K.

5. By adding KHG to each equal, prove the adjacent angles at H are equal to two right angles.

6. Draw the inference from this, giving the reference.

7. Prove that the angles FGH and L are equal; given, as before, that the angle E is equal to each of the angles K, L.

8. Add HGL to each of these equals, and then proceed to prove FGL one straight line.

9. If FH, HL are parallelograms, prove FK, LM are parallels.

10. If ABC be any triangle, having BC for base, and D is taken bisecting AB, and then joined to C, show that the triangle ABC is double of the triangle DBC.

11. Given ABC any triangle and BC its base; apply to BC a parallelogram equal to ABC, and having an angle equal to a given rectilineal angle, without making use of Prop. XLIII.

12. Given a triangle; apply to any side of it a rectangle equal in area to the triangle, without applying XLIII.

98. Proposition XLV.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let $ABCD$ be the given rectilineal figure,
and E the given rectilineal angle.

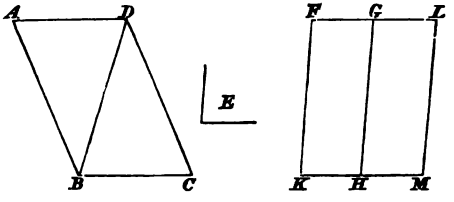
It is required to describe a parallelogram,
that shall be equal to ABCD, and have an angle equal to E.

Join *DB*.

Describe the parallelogram *FH* equal to the triangle *ADB*, and having the angle *FKH* equal to *E*; (I. 42)

to the straight line *GH* apply the parallelogram *GM* equal to *DBC*, and having the angle *GHM* equal to *E*. (I. 44)

Then the figure FKML shall be the parallelogram required.



Because *E* is equal to each of the angles *FKH*, *GHM*,
therefore *FKH* is equal to *GHM*;
add to each the angle *KHG*;
therefore the angles *FKH*, *KHG* are equal to *KHG*, *GHM*;
but *FKH*, *KHG* are equal to two right angles; (I. 29)
therefore, also, *KHG*, *GHM* are equal to two right angles,
that is, at the point *H* in the straight line *GH*, the two *KH*, *HM*, upon the opposite sides of it, make the adjacent angles equal to two right angles;

therefore HK is in the same straight line with HM. (I. 14)

And because *HG* meets the parallels *KM*, *FG*,
the angle *MHG* is equal to the alternate angle *HGF*; (I. 29)
add to each the angle *HGL*;
therefore *MHG*, *HGL* are equal to *HGF*, *HGL*;
but *MHG*, *HGL* are equal to two right angles; (I. 29)
therefore, also, *HGF*, *HGL* are equal to two right angles;

wherefore FG is in the same straight line with GL. (I. 14)

Again, because *KF* is parallel to *HG*, and *HG* to *ML*,
therefore *KF* is parallel to *ML*; (I. 30)
and *KM* has been proved parallel to *FL*;

therefore the figure FKML is a parallelogram.

And since FH is equal to the triangle ABD,
and GM to the triangle BDC;
therefore the whole $FKML$ is equal to the whole $ABCD$.
Wherefore a parallelogram $FKML$ has been described equal to the given rectilineal figure $ABCD$,
and having the angle FKM equal to the given angle E.
Q. E. F.

COR.—From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given angle, and shall be equal to a given rectilineal figure of any number of sides;

namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle (I. 44); and then applying to a side of this parallelogram another equal to the second triangle; and so on.

99. REDUCTION OF A POLYGON TO A TRIANGLE OF EQUIVALENT AREA.

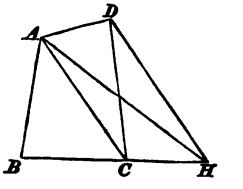
By means of Props. XLIV. and XLV., Euclid reduces any polygon to a parallelogram of equivalent area. It is considerably easier to reduce a polygon to a triangle of the same area; and, as the method is of practical utility, it will now be given.

PROBLEM.—*To construct a triangle equal in area to a given rectilineal figure.*

Let $ABCD$ be the given figure.

It is required to construct a triangle equal to $ABCD$.

Join AC;
draw DH parallel to AC, (I. 31)
meeting BC produced in H;
and join AH.



Then ABH is the triangle required.

Because the triangles HAC, DAC are upon the same base AC, and between the same parallels DH, AC;
therefore HAC is equal to DAC. (I. 37)
To each equal add the triangle ABC;
then the whole triangle HAB is equal to the whole figure $ABCD$.

Wherefore, a triangle ABH has been constructed equal in area to the given rectilineal figure $ABCD$. Q. E. F.

Continuing the preceding, we may take a parallelogram equal to the tri-

angle ABH (I. 42); and the whole proposition would then be equivalent to I. 45.

In the proof given, just as in Euclid, the rectilineal figure is taken quadrilateral. If there be another side to the figure, it must be divided into three triangles by drawing straight lines from one angle; and for every additional side there will be an additional triangle.

The process would then be to replace two adjacent triangles by one, as just shown; then replace this new one and an adjacent triangle in the same way; and so on till all are reduced.

100. Exercise LIII.

1. Make a figure to XLIII., having K the middle point of AC; then show that the triangles AHK, KFC are equal by I. 26.

2. Also prove that under the same circumstances the parallelograms about AC are equal.

3. Join HF, in the figure just referred to, and then prove HF parallel to AC.

4. Also join EG, and prove it parallel to HF.

5. In the usual figure to XLIII., join EG and HF; and prove the triangles EGK, FHK equal.

6. If FG be joined and the triangle KFG be added to each equal, what is the resulting pair of equals?

7. Now join EH, and prove EH parallel to GF.

8. Let EH and GH cut AC in L, M respectively; prove LM to be half of AC.

9. Given a triangle and a straight line, make a rectangle upon the straight line equal to the triangle.

10. Given a triangle and a straight line, make a rectangle upon the straight line double of the triangle.

11. Show how to make a parallelogram double of a given rectilineal figure, having an angle equal to two-thirds of a right angle.

12. Describe a rhombus equal to a given rectilineal figure.

SECTION XI. PROPOSITIONS XLVI—XLVIII.

Squares and the Sides of a right-angled Triangle.

101. Proposition XLVI.—Problem.

To describe a square upon a given straight line.

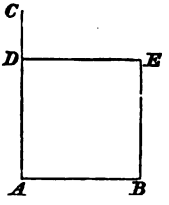
Let AB be the given straight line.

It is required to describe a square upon AB.

From the point A draw AC at right angles to AB; (I. 11)

make AD equal to AB; (I. 3)
through D draw DE parallel to AB, (I. 31)
and through B draw BE parallel to AD.

Then $ABED$ shall be the square required.



Now $ABED$ is a parallelogram; (Constr.)
therefore AB is equal to DE, and AD to BE; (I. 34)
but AB is equal to AD; (Constr.)
therefore BA, AD, DE, EB are all equal to one another,
and the parallelogram $ADEB$ is equilateral.

Again, since AD meets the parallels AB, DE,
the angles BAD, ADE are equal to two right angles; (I. 29)
but BAD is a right angle; (Constr.)
therefore also ADE is a right angle;
therefore each of the opposite angles ABE, BED is a right angle; (I. 34)

wherefore the figure $ADEB$ is rectangular.

And it has been proved to be equilateral;
therefore it is a square. (Def. 30)

Wherefore a square $ADEB$ has been described upon the given straight line AB. Q. E. F.

Cor.—Every parallelogram that has one right angle has all its angles right angles.

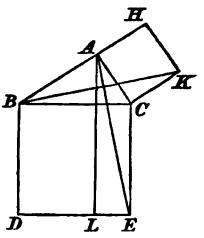
102. Exercise LIV.

1. Where is the proposition found in Euclid—that, if a parallelogram has one angle a right angle, it is a rectangle or oblong?

2. If a parallelogram were constructed with two adjacent sides equal, and one angle right, what would it become?

What proposition has its construction identical with this?

3. Draw a figure like that of XLVI., and join CB, CE; then prove that the triangle CBE is half the square AE.



4. Given, in the adjacent figure, BAC a right angle and CH a square; what is the sum of the adjacent angles at A?

5. Draw an inference by I. 14.

6. Given, also, that BE is a square; prove the angles BCE, ACK equal.

7. Thence prove angles ACE, BCK equal.

8. Also show that EC, CA are equal to BC, CK, each to each.

9. From the results of 7 and 8, find two triangles equal by I. 4.

10. Prove, in the same figure, BH and CK are parallel.

11. Given also that AL is parallel to CE, find parallelograms respectively double of the triangles ACE and BCK.

12. Draw an inference by Ax. 6.

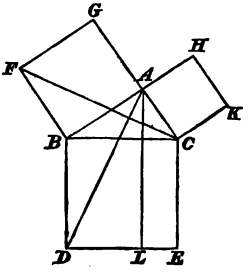
103. Proposition XLVII.—Theorem.

In any right-angled triangle, the square which is described upon the side subtending the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle,
having BAC the right angle.

Then the square described upon the side BC
shall be equal to the squares described upon BA, AC.

On BC describe the square $BDEC$, (I. 46)
and on BA, AC the squares GB, HC;
through A draw AL parallel to BD or CE; (I. 31)
and join AD, FC.



Then because the angle BAC is a right angle, (Hyp.)
and that BAG is also a right angle, (Def. 30)
the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A the adjacent angles together equal to two right angles;
therefore CA is in the same straight line with AG. (I. 14)
For the same reason, BA and AH are in one straight line.

Now the angle DBC is equal to the angle FBA,
each of them being a right angle;
add to each the angle ABC;
then the whole angle DBA is equal to the whole FBC. (Ax. 2)

And because the two sides AB, BD are equal to the two FB, BC, each to each,
and the included angle ABD is equal to the included angle FBC;
therefore the triangle ABD is equal to the triangle FBC. (I. 4)

Now the parallelogram BL is double of the triangle ABD (I. 41)
because they are upon the same base BD, and between the same parallels BD, AL;
also the square GB is double of the triangle FBC,
because they are upon the same base FB, and between the same parallels FB, GC;
therefore BL is equal to the square BG. (Ax. 6)

Similarly, by joining AE, BK, it can be proved
that CL is equal to the square HC.

Therefore the whole square $BDEC$ is equal to the two squares GB, HC; (Ax. 2)
that is, the square upon the side BC is equal to the squares upon the sides AB, AC.

Therefore, in any right-angled triangle, &c.

Q. E. D.

104. Exercise LV.

1. Give fully the proof showing that the parallelogram CL and the square CH are equal.

2. Prove that the square upon AD, in the figure to XLVII., is equal to the squares on AL, LD.

3. If CD be joined in the same figure, the square upon CD is equal to the squares upon AB, BC, CA.

4. In any right-angled triangle ABC, having C the right angle, the square upon BC is equal to the difference between the squares on AB and AC.

5. The square described upon the diagonal of a square is equal to twice the latter square.

6. In a rectangle the sum of the squares on the two diagonals is equal to the sum of the squares on the sides.

Note.—Proposition XLVII. is often called the Theorem of Pythagoras, from the name of its discoverer, an ancient Greek philosopher, who lived in the sixth century before Jesus Christ.

It has important uses in arithmetical calculations, as well as in Geometry. If the learner could practice some of these, while studying this part of Geometry, it would be an advantage.

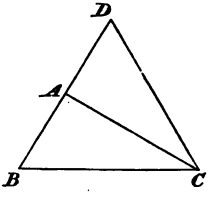
105. Proposition XLVIII.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC,

be equal to the squares upon the other two sides AB, AC.

Then the angle BAC shall be a right angle.



From the point A draw AD at right angles to AC; (I. 11) make AD equal to AB, and join DC.

Then, because AD is equal to AB,

the square upon AD is equal to the square upon AB;

to each of these equals add the square on AC;

then the squares on AD, AC are equal to the squares on AB, AC.

But the squares on AD, AC are equal to the square on DC, (I. 47) because DAC is a right angle;

and the squares on BA, AC are equal to the square on BC; (Hyp.)

therefore the square on DC is equal to the square on BC;

wherefore the side DC is equal to the side BC.

And because AD is equal to AB,

and AC is common to the triangles DAC, BAC;

therefore DA, AC are equal to BA, AC, each to each;

and the base DC has been proved equal to the base BC;

therefore the angle DAC is equal to BAC. (I. 8)

But DAC is a right angle, (Constr.)

therefore also BAC is a right angle,

Therefore, if the square, &c.

Q. E. D.

106. Exercise LVI.

1. State the relation of XLVIII. to XLVII.

2. If the square on one side of an isosceles triangle be half the square on another, prove the triangle is also right-angled.

3. Given two squares, show how to find a square equal in area to their sum.

4. Given a square, show how to find a square which is its double in area.

5. Given a square, find another of three times its area.

6. Given three squares, find another whose area is equal to the sum of the three.

7. Show that the square on the diagonal of a cube is three times one face of the cube.

8. State the proposition of which the enunciation may be thus expressed:

"Construct an equilateral rectangle, each side of which shall be equal to a given straight line."

9. Give the proposition equivalent to:

"The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares upon the other two sides."

10. In the construction to XLVIII., show that it would be unsatisfactory to say, instead of the first line,

"Produce BA to D,"

unless we choose to add a short piece of reasoning to justify it.

Give the reasoning which would be required.

11. State the assumption made in the foregoing proof where the inference is drawn:

"*Wherefore the side DC is equal to the side BC.*"

EXAMINATION PAPERS FOR REVISION.

NOTE 1.—These papers are suitable for general purposes; but have been constructed with a view to being worked during a revision of Euclid's text, as given in the larger print. Section I. here might then come after Section I. of the text; and similarly for the other sections.

NOTE 2.—The student may be forewarned that a few of the following exercises consist of propositions of Euclid enunciated in different language to Euclid's own. A larger number consist of portions of Euclid's propositions; and in each of these cases the proof should be given only so far as required by the portion which is set.

SECTION I. PROPOSITIONS 1—6.

I.

1. With a given straight line as one side, construct a rhombus, which shall be composed of two equilateral triangles.

2. X, Y, Z are three points in a straight line; from Z draw a straight line equal to XY, using the construction of Prop. II.

3. Given AB and C two straight lines, of which AB is the greater; to cut off from BA a part equal to C, taking it from the B end of the line, after the manner of Prop. III.

4. If two triangles have two sides of the one and the angle which they contain respectively equal to two sides of the other and the angle which these contain, then the triangles shall be equal in area.

5. If two sides of a triangle are equal to one another, and these sides be produced, the two *exterior* angles of the triangle thus formed are equal to one another. (The exterior angles are those *outside* the triangle, "on the other side of the base.")

6. ABC is a triangle on a level plane, having the angles B and C equal; we require the distance from C to A, but cannot measure it; while we *can* measure either of the other sides of the triangle. What shall we do?

II.

1. XYZ is a straight line; from X draw a straight line equal to YZ, using the construction of Prop. II.

2. Given AB and C two straight lines; lengthen AB by a Euclidian method so as to make it equal to AB and C taken together.

3. If two sides and the included angle of one triangle be equal to two sides and the included angle of another, prove the third sides of the triangles are equal.

4. If C be the middle point of AB, and CD be at right angles to AB, prove that DAB is isosceles.

5. If FG, in the figure to Proposition V., be joined, then the angles AFG and AGF shall be equal.

6. With the same construction, show that the angles CFG and BGF will be equal.

III.

1. XYZ is a straight line; from Y draw a straight line equal to XZ, using the construction of Prop. II.

2. Given a straight line AB, make an isosceles triangle upon it with each of the equal sides twice as long as AB.

3. If two triangles have two sides and the contained angle of the one equal to two sides and the contained angle of the other, each to each, they shall have their remaining angles equal in pairs; each of these pairs of angles having a pair of equal sides opposite to it.

4. In an isosceles triangle, show that the right line which bisects the vertical angle bisects the base, and is perpendicular to the base.

5. In an isosceles triangle the line joining the vertex to the middle point of the base is perpendicular to the base, and bisects the angle at the vertex.

6. If the angle ABC of the triangle ABC be double of the angle C, and BD bisect the angle ABC, meeting AC in D, show that DBC is isosceles.

IV.

1. P is a point and QR a straight line; from P draw a straight line twice as long as QR.

2. Two right-angled triangles are equal, if they have the sides containing the right angle in one respectively equal to the sides containing the angle in the other.

3. The angles A, B, C of a triangle are respectively 50°, 80°, 50°; which sides are equal?

4. If C be the middle point of AB, the base of the triangle DAB, and DC be perpendicular to AB, then the angle ADB is bisected by DC.

5. Every quadrilateral whose diagonals bisect each other at right angles is a rhombus.

6. If the angles ABC, ACB at the base of an isosceles triangle be bisected by two straight lines BD, CD; then DBC will be isosceles.

V.

1. ABC is an equilateral triangle, and D, E, F are points in AB, BC, CA respectively, such that AD, BE, CF are all equal; if D, E, F be joined, the triangle thus formed will be equilateral.

2. Show that, if two straight lines bisect one another at right angles, any point in one of them is equidistant from the extremities of the other.

3. Prove that the straight lines which bisect two sides of a triangle at right angles meet in a point which is equidistant from the three angles.

4. In the figure to Euclid I. 5, if BG, CF intersect in H, and AH be joined, show that it bisects the angle A. (First show HB, HC equal by I. 6; then apply I. 4 to the triangles ABH, ACH.)

5. Deduce a construction for bisecting a given rectilineal angle, which requires no proposition of Euclid beyond I. 3; and whose proof requires no proposition beyond I. 6.

6. $ABCD$ is a quadrilateral, having its four angular points on the circumference of a circle; show that the sum of one pair of opposite angles is equal to the sum of the other pair. (Join A, B, C, D to the centre, and use I. 5 four times.)

SECTION II. PROPOSITIONS 7—12.

VI.

1. If upon the same base and on the same side of it there be two triangles, and one of them be equilateral, prove that the other is not equilateral.

2. In equal circles, equal chords subtend equal angles at the centres.

3. In what particular case will the quadrilateral in the figure of I. 9 be a rhombus?

4. Show how to find a point in side AB of any triangle ABC which is equidistant from the angles B and C.

5. What postulate should be applied in order to make the construction of I. 11 available when the given point is an extremity of the given line?

6. If a straight line be drawn from the vertex of an equilateral triangle to the middle point of the base, prove that every point in it is equidistant from each base angle. (Apply I. 8 and I. 4 successively.)

VII.

1. Two triangles are on opposite sides of a common base, and their other sides are respectively equal, and also one pair of equal sides are in one straight line; prove that each triangle is right-angled. (Use I. 8 and Def. 10.)

2. If two quadrilaterals have the four sides of the one respectively equal to those of the other in order, and the angle contained by two sides of one ›ual to the angle contained by the two sides equal to them of the other,

then all the remaining angles of the figure shall be equal in pairs. (First prove two diagonals equal by I. 4, then use I. 8.)

3. Construct a right-angled triangle, given one side and the hypotenuse.

4. A is a given point, and BC a given straight line. Let the straight line which bisects AB at right angles meet BC in D. Show that D is equally distant from A and B.

5. A and B are given points above a given straight line CD; if there be a point in CD equally distant from A, B, show how to find it.

6. If two circles cut each other, the straight line joining the two points of intersection is at right angles to the line which joins their centres.

VIII.

1. Given a finite right line of any length; bisect it, and at the point of bisection erect a perpendicular to it.

2. Having given two points, find two others which shall be at a given distance from each of the first two.

When will the problem become impossible?

3. AB is a given straight line, and C, D are given points on opposite sides of AB; if there be a point in AB equally distant from C, D, find it.

4. If a given point be above a given straight line, but towards one end of it, and we do not wish to produce the line, prove that the following construction will give a perpendicular from the point to the line:—

> Take any two points along the line; with each as centre, at the distance in each case of the given point, draw two arcs, meeting twice; join these two intersections of the arcs.

5. If BDC be drawn perpendicular to DA, and DC be made equal to BD; then two straight lines drawn from B and C to any one point in DA will be equally inclined to DA.

6. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by it.

SECTION III.—PROPOSITIONS XIII.—XVII.

IX.

1. If two straight lines meet at a point, and form there an angle equal to three-fourths of a right angle, what would be the magnitude of the adjacent angle which would be formed by producing through the point either of the straight lines?

2. Four straight lines are drawn from one point, and three successive angles thus formed are respectively two-thirds, seven-sixths, and five-

sixths of a right angle in magnitude; show that two of the straight lines coincide, and that the other two do not.

State the magnitude of the fourth angle.

3. The straight lines which bisect opposite vertical angles are in one and the same straight line.

4. Prove that any two exterior angles of a triangle are together greater than two right angles.

5. Show that only one perpendicular can be drawn to a given straight line from a given point without it.

6. Prove Euclid's Proposition XVII. without producing any side, but using a straight line joining any vertex to a point in the opposite side.

X.

1. If four straight lines meet at a point, so as to make the opposite angles at that point equal, these straight lines are two and two in the same straight line.

2. If CD make with AB the adjacent angles ACD, DCB, prove that straight lines drawn bisecting these angles will contain a right angle.

3. From a given point let a straight line be drawn to a given straight line, making two angles with it; then, if one of these be acute, prove that the other will be obtuse.

If from the point just mentioned a perpendicular be also drawn to the same given straight line, prove that it will subtend that angle of the two which is acute.

4. The supplement of any angle of a triangle is greater than either of the other two.

5. From the same point outside a given straight line it is not possible to draw to it three straight lines which are all equal.

6. Two finite right lines, of any lengths, being supposed to radiate in any directions from a common point; show that the angle they determine is equal to that determined by their two productions through the point.

SECTION IV.—PROPOSITIONS XVIII.—XXV.

XI.

1. In an obtuse-angled triangle, show that the longest side is opposite the obtuse angle.

2. If the base is the least side of an isosceles triangle, prove that the triangle is acute-angled.

3. Any three sides of a quadrilateral are greater than the fourth side.

4. If two triangles be upon the same base, but the vertex of one within the other triangle, then the perimeter of the inner triangle is less than that of the other. [Perimeter means sum of the sides.]

5. Two sides of a triangle being supposed given in magnitude, show by any method that the greater the angle between them, the greater the third side.

6. If one angle of a triangle be equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

XII.

1. ABC is a triangle, and the angle A is bisected by a straight line which meets BC in D. Show that BA is greater than BD, and CA greater than CD.

2. If a straight line be drawn through A, one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced in F, show that AF is greater than the diagonal of the square.

3. The two sides of any triangle are together greater than twice the straight line drawn from the vextex to the middle of the base.

4. Two equal circles have an angle at the centre of each; if these are unequal, the chord opposite the greater angle is greater than that opposite the other.

5. In two equal circles, having a chord in one greater than a chord in the other, show that the angle subtended at the centre of its circle by the first chord is greater than that similarly subtended by the second.

6. Show that three rods pivoted together at their extremities will form a rigid triangle; that is, no two sides will work about the pivot at their common extremity.

XIII.

1. From the vertex A of an isosceles triangle, two straight lines are drawn, one to meet the base BC in D, and the other to meet BC produced in E; show that AB is less than AE, and greater than AD.

2. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides.

3. If the vertical angle of a triangle be equal to the sum of the other two, the straight line drawn from the vertex to the middle of the base will be equal to half the base. [See XI. 6.]

4. $ABCD$ is a quadilateral, of which AD is the longest side and BC the shortest; show that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD. (Join BD to begin with.)

5. Prove Euclid I. 18, given AC greater than AB in the triangle ABC, by cutting AD off AC equal to AB, and joining D to E, where E is the point in which the bisector of the angle A meets BC.

6. I have a quadrilateral framework of four rods jointed at their extremities; prove that it will become rigid if I fasten to two opposite corners of it the two ends of another rigid rod.

XIV.

1. If two chords be drawn from one point in the circumference of a circle, one of them passing through the centre, this latter will be longer than the other.

2. The four sides of a quadrilateral are together greater than the sum of its diagonals.

3. If straight lines be drawn from a point within a triangle to each angle, the sum of them shall be less than the perimeter of the triangle.

4. Construct a triangle, given the base, an angle at the base, and the sum of the two sides.

5. Show that, of the two diagonals of a parallelogram which is not a rectangle, the greater connects the two acute, and the lesser the two obtuse angles of the figure.

6. Of all the straight lines which can be drawn from a given point to a given straight line—

(*a*) The perpendicular is less than any other;

(*b*) Any two which are equidistant from the perpendicular are also equal.

(*c*) Any one which is nearer the perpendicular than another is also less than it.

From (*b*) and (*c*) infer that not more than two of the lines can be equal to one another.

SECTION V.—PROPOSITION XXVI.

XV.

1. Prove that a triangle is isosceles if the bisector of any angle is perpendicular to the opposite side.

2. If a straight line be drawn from the vertical angle of an isosceles triangle perpendicular to the base, it shall bisect the base.

3. The same straight line shall bisect the vertical angle.

4. If from any point in the same perpendicular two straight lines be drawn to the ends of the base, they will form with the base an isosceles triangle.

5. If a straight line be drawn bisecting an angle, any point in it is equidistant from the two straight lines which contain the angle.

6. A straight line terminated by two parallel straight lines is bisected at a point C. Prove that every straight line through C which is terminated by the parallels is bisected at C.

XVI.

1. Given an angle BAC and a point D; show that a perpendicular from D upon the bisector of the angle A will cut off equal distances along AB, AC.

2. Through a given point draw a straight line which shall be equally inclined to two given straight lines.

3. Find a point in the base of any given triangle which is equidistant from the sides.

4. Two right-angled triangles are equal in all respects if the hypotenuse and an acute angle of one are respectively equal to similar parts of the other.

5. From a given point draw a straight line which shall be equidistant from two other given points.

6. From any point in an isosceles triangle, which is not in the bisector of its vertical angle, draw straight lines to the ends of the base; and show that these are not equal.

Is the new triangle thus formed necessarily scalene?

SECTION VI. PROPOSITIONS XXVII.—XXXI.

XVII.

1. Through a given point on either side of a given indefinite right line draw the perpendicular and the parallel to the line.

2. A parallel to the base BC of a triangle ABC cuts the sides, produced if necessary, at D and E. Prove that the new triangle ADE has its angles equal to those of the original triangle ABC.

3. Show that every parallel to the base of an isosceles triangle cuts off equal lengths on the sides measured from the vertex of the triangle.

4. If a straight line be drawn parallel to one side of a parallelogram through an internal point, it divides the figure into two parallelograms.

5. Two rectilinear segments of any lengths being supposed to have a common middle point, but not a common direction, show that they are the two diagonals of a parallelogram.

6. Show how to construct a triangle having its vertex at a given point, its base on a given indefinite right line, and its base angles equal to two given angles.

XVIII.

1. Let ABC be any triangle; at A, and on the side of AB remote from C, make an angle BAE equal to ABC; then prove that a parallel has been drawn to BC.

2. In the triangle ABC, having AB produced to D, make at B, on the same side of DB as C, an angle DBF equal to BAC; then BF will be parallel to CA.

3. In the triangle ABC, make at C, on the same side of BC as A, an angle BCH equal to the supplement of ABC; then CH will be parallel to AB.

4. If a quadrilateral have two adjacent angles equal to two right angles, it will also have two sides parallel.

5. If a straight line be parallel to the base of an isosceles triangle, it is equally inclined to the two sides or the two sides produced.

6. If a parallelogram be formed out of bars pivoted together at the angles, when one angle is opened out, show that one diagonal of the figure is increased and the other diminished.

XIX.

1. Two intersecting straight lines which are terminated in two parallel straight lines shall bisect one another, if they cut off equal portions along those parallels.

2. If a straight line be terminated by two parallels, any other straight line passing through its middle point will be bisected there, if its ends be in the parallels.

3. If two straight lines be respectively parallel to two others, the angle between the first two will be equal to the angle between the other two.

4. If the straight line which bisects the exterior angle at the vertex of a triangle be parallel to the base, the triangle is isosceles.

5. If an angle be bisected, the two straight lines drawn from any point in the bisector, each parallel to one side of the angle and meeting the other, will be equal.

6. A straight line drawn at right angles to the base of an isosceles triangle cuts one side in P, the other produced in Q. If R be the vertex of the original triangle, show that the triangle PQR is isosceles also.

SECTION VII. PROPOSITION XXXII. AND COROLLARIES.

XX.

1. A triangle is obtuse-angled if one angle be greater than the sum of the other two, right-angled if equal, and acute-angled if each angle be less.

2. A straight line which makes equal angles with two sides of an isosceles triangle will be parallel to the base.

3. In an isosceles triangle, each angle at the base is equal to half the exterior angle at the vertex.

4. If two triangles have the sum of two angles in one triangle equal to the sum of two angles in the other, their third angles are also equal.

5. Each angle of an equilateral triangle is two-thirds of one right angle. Apply this in trisecting a right angle.

6. Determine an exterior and an interior angle of a regular pentagon.

XXI.

1. A triangle is right-angled if the straight line drawn from the vertex to the middle point of the base be equal to half the base.

2. The vertical angle of the same triangle is acute if this line be greater than half the base; but obtuse if less.

3. If the sides of a regular pentagon be produced to meet in five points, the angles at these points are together equal to two right angles.

If the figure were a hexagon, the angles would be equal to four right angles.

4. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle, is equal to half the hypotenuse.

5. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

6. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

XXII.

1. Prove that no polygon of the ordinary form can have more than three of its exterior angles obtuse, or more than three of its interior angles acute.

2. Construct an equilateral triangle having its altitude given.

3. Construct an isosceles triangle whose base and angle at the vertex are given.

4. Two of the angles in a triangle, which is not given, are known to be equal to two given acute angles. Show how to make, with a given straight line, an angle which shall be equal to the third angle in the triangle.

5. Show how to construct a triangle, having given two angles and the side opposite one of them.

6. If one diagonal of a quadrilateral be diminished without altering the lengths of the sides, the other diagonal will be increased.

SECTION VIII. PROPOSITIONS XXXIII., XXXIV.

XXIII.

A quadrilateral figure will be a parallelogram—

1. If two opposite sides be both equal and parallel; *or*
2. If its opposite sides are equal; *or*
3. If its opposite angles are equal; *or*
4. If its diagonals bisect one another.
5. If a quadrilateral be a parallelogram, its diagonals bisect one another.
6. Through a given point to draw a straight line cutting two parallel straight lines so that the portion intercepted between them shall be equal to a given straight line.

XXIV.

1. If each diameter of a quadrilateral bisect two opposite angles, it must be a rhombus.
2. If the diagonals of a parallelogram are equal, all its angles are equal and right.
3. In a parallelogram, of any magnitude and form, show that each of the four angles, and also each of the four sides, is equal to its opposite.
4. By aid of I. 34 or otherwise, draw a chord of a given angle, which shall be at once parallel and equal to a given finite right line.
5. Given two straight lines which meet, to find a point such that the perpendiculars let fall from it upon the two given lines shall be respectively equal to two other given straight lines.
6. Construct a parallelogram which shall have two adjacent sides and the diagonal through their point of intersection equal to three given straight lines of finite lengths.

SECTION IX. PROPOSITIONS XXXV.—XLII.

XXV.

1. A parallelogram being given which is not rectangular, construct a rectangle equal to it.
2. Show how to bisect a triangle by a straight line drawn through the vertex.
3. If a quadrilateral figure be bisected by each diagonal, it is a parallelogram.
4. A straight line drawn through the middle point of the diameter of a parallelogram, and terminated by two opposite sides, bisects the parallelogram.

5. The four triangles into which a parallelogram is divided by its diagonals are equal.

6. If, in the figure of I. 42, the triangle be equilateral, and the given angle two-thirds of a right angle, the perimeters of the parallelogram and the triangle are also equal.

XXVI.

1. A parallelogram being given which is not equilateral, construct a rhombus equal to it in area.

2. $ABCD$ is a quadrilateral, having BC parallel but not equal to AD; a parallelogram is formed by drawing through the middle of DC a straight line parallel to AB to meet the parallel sides, produced if necessary; show that this figure is equal to the original quadrilateral.

3. If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles contained by those sides supplementary, the triangles are equal in area.

4. Bisect a parallelogram by a straight line drawn through any given point outside of it.

5. If from a point in one diagonal of a parallelogram straight lines be drawn to the opposite angles, these will form two pairs of equal triangles.

6. Describe a parallelogram equal to a given triangle, and such that the sum of its sides shall be equal to the sum of the sides of the triangle.

XXVII.

1. Make two parallelograms which shall have their sides equal, each to each, but the area of one double that of the other.

2. If $ABCD$ be a quadrilateral whose diagonals meet in E, and the triangles AED, BEC are equal; show that AB and CD are parallel.

3. The straight line which joins the middle points of two sides of a triangle is parallel to the base and equal to half of it.

4. Straight lines which join the middle points of adjacent sides of a quadrilateral form a parallelogram.

5. If the middle points of the sides of a triangle be joined, the triangle is divided into four equal triangles.

6. If straight lines are drawn from any point within a parallelogram to the angles, the two triangles thus formed, upon either pair of opposite sides of the figure as bases, are together equal to half the whole figure.

XXVIII.

1. Prove that a parallelogram is equal in area to a rectangle which has its base and altitude equal to the base and altitude of the parallelogram.

2. Two triangles, having a common base, being supposed to have their two vertices in a common parallel to the base; show that the four

parallelograms on the same base, having their four sides for diagonals, are equal in area.

3. By aid of the preceding, or otherwise, construct on a given base a triangle of given area, having its vertex on a given indefinite right line not parallel to the base; and determine the number of solutions.

4. If through the middle point of a side of a triangle a straight line be drawn parallel to the base, it will bisect the other side, and that part of it which is intercepted between the sides of the triangle will be equal to half the base.

5. Prove that a quadrilateral which has two opposite sides parallel is equal in area to a rectangle between the same parallels on a base equal to half the sum of the parallel sides of the given quadrilateral.

6. On a given finite right line of any length as base, construct a rectangle which shall be equal to a given square.

SECTION X. PROPOSITIONS XLIII.—XLV.

XXIX.

1. The parallelograms about the diagonal of a square are likewise squares.

2. To one of the sides of an equilateral triangle apply an equal parallelogram having one of its angles equal to that of the triangle.

3. Construct a parallelogram equal to the difference of two rectilineal figures.

4. If through a point O within the parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms OB, OD are equal, the point O is in the diagonal AC.

5. To a given straight line apply a triangle equal to a given parallelogram, and having an angle equal to a given rectilineal angle.

6. Describe a triangle upon a given base equal in area to a given triangle.

SECTION XI. PROPOSITIONS XLVI.—XLVIII.

XXX.

1. Squares constructed upon equal straight lines are equal themselves, and conversely.

2. Describe a square equal to the double of a given square.

3. The square upon a given straight line is equal to four times the square upon its half.

4. If a perpendicular be drawn from the vertex upon the base of any triangle, the difference between the squares on the sections of the base is equal to the difference between the squares on the two sides of the triangle.

5. Construct a square equal to the difference between two given squares.

6. Prove that the square on the side subtending an acute angle of a triangle is less than the square upon the sides containing that angle.

XXXI.

1. Prove that two right-angled triangles are equal in all respects if the hypotenuse and a side in the one are respectively equal to the hypotenuse and a side in the other.

2. Describe a square equal to three times a given square; also a square equal to the sum of three given squares.

3. The squares upon the sides of a rhombus are together equal to the squares upon its diagonals.

4. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments so formed is equal to the square on the other side.

5. If a straight line be divided into any two parts, the square upon the whole line is greater than the sum of the squares on the parts.

6. The square upon the side subtending an obtuse angle of a triangle is greater than the sum of the squares upon the sides which contain the obtuse angle.

SCIENCE AND ART DEPARTMENT'S PAPERS,

Set in the Examinations in Pure Mathematics, Stage I.

NOTE.—The numbering of the questions is given as in the actual papers; in which six questions in Arithmetic come before these in Geometry, and six in Algebra come after.

The numbers in brackets are those assigned by the examiner, and indicate the relative importance of the questions.

1875.

7. Prove that the angles at the base of an isosceles triangle are equal to one another. (8)

8. Prove that any two sides of a triangle are together greater than the third. (8)

9. Let A, B, C be three points in a straight line taken in order, and D

any other point not in that line. Join BD. Show that the lines which bisect the angles ABD and DBC are perpendicular to one another. (10)

10. Show that a line drawn from the angle (or vertex) of a triangle so as to bisect the base, also bisects the triangle. (12)

11. Prove that the diagonals of a parallelogram bisect one another. (14)

12. Show that, if a straight line meets two parallel straight lines, the alternate angles are equal. Write out at full length what definition of parallelism, and what axiom (if any), you use to prove this. *No marks will be given for a mere proof if this be not done.* (14)

1876.

7. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal. (8)

8. Any two angles of a triangle are together less than two right angles. (8)

9. The opposite sides and angles of a parallelogram are equal to one another. (9)

10. If two isosceles triangles have the same base, the line which joins their summits is at right angles to the base. (12)

11. ABC is an equilateral triangle. On AB take any distance AF less than AB. On BC take BD equal to AF, and on CA take CE equal to AF. Join AD, BE, and CF. Show that these three lines either meet in one point, or enclose an equilateral triangle. (14)

12. What use does Euclid make in his First Book of the tenth axiom, namely, that two straight lines cannot enclose a space? (14)

1877.

7. Prove that the three angles of a triangle, taken together, are equal to two right angles. (8)

8. Through a given point outside a straight line draw another straight line which shall make with the first an angle equal to a given angle.

How many such lines can be drawn? (9)

9. Prove that, if the squares upon two of the sides of a triangle, taken together, are equal to the square on the third side, the triangle is right-angled.

Which side is the right angle opposite to? (9)

10. Prove that, if the four sides of a parallelogram are all equal, the two diagonals bisect, and are perpendicular to, one another. (12)

11. The equal sides of an isosceles triangle are each 13 inches long, and the base is 10 inches. Find the length of the shortest line which can be drawn from the vertex to the base. (12)

12. What do you understand by the equality of two angles?

By what geometrical process would you ascertain which was the greater of two angles actually drawn upon a sheet of paper (their vertices not coinciding)? (14)

1878.

7. Prove that a triangle cannot have two obtuse angles. (8)

8. Prove that the two angles which one right line makes with another (on the same side of either line) are either two right angles, or are together equal to two right angles. (8)

9. Prove that the complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another. (10)

10. $ABCD$ is a trapezium, of which the side AB is parallel to the side CD, and its diagonals AC and BD meet in O. Prove that the triangles AOD and BOC are equal. (12)

11. Show how to construct a rectangle (of which one side is given) which shall be equal to a given rectangle, and prove your construction. (14)

12. The side BC of a triangle ABC is bisected in D. DF is drawn parallel to CA to meet AB in F, and DE is drawn parallel to AB to meet CA in E. Join EF.

Show that the four triangles AEF, BFD, CDE, DEF are equal in all respects. (14)

1879.

7. Define parallel straight lines. Show how to draw through a given point a straight line parallel to a given straight line, and prove your construction. (8)

8. Prove that any two sides of a triangle, taken together, are greater than the third side.

If the distance between the centres of two circles exceeds their radii, taken together, prove that the circles will not meet one another. (8)

9. O is the point of intersection of the diagonals of a square, and through it are drawn any two straight lines at right angles to one another, meeting the sides in E, F, G, H, and the sides produced in P, Q, R, S. Prove that $EFGH$ and $PQRS$ each form a square. (10)

10. ABC is a triangle right-angled at C, and CD is drawn at right angles to AB. Show that the square on AB exceeds the squares on AD and DB by twice the square on CD. (10)

11. A frame consists of two equal bars, AB and CD, connected by two other equal bars, AD and BC, which cross one another, and which move freely on the pivots A, B, C, D. Show that the two triangles formed by the frame, in any position, are equal in all respects. (14)

12. Show that, if one side of a triangle be produced, the exterior angle thus formed is equal to the two interior opposite angles.

Given one side of a triangle, one angle adjacent to it, and that the other adjacent angle is three times the third angle, construct the triangle. (14)

1880.

7. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, show that the angle contained by the two sides of the one triangle is equal to that contained by the two sides of the other. (16)*

8. If a quadrilateral figure is divided into two isosceles triangles by one diagonal, show that the diagonals are at right angles to each other. (8)

9. Two circles have the same centre C. Two points A, B are taken on one circle, and two points P, Q on the other, so that the angles ACP and BCQ are equal. Show that the triangles ACP, BCQ are equal in all respects. (10)

10. Prove that triangles on equal bases and between the same parallels are equal.

$ABCD$ is a parallelogram, of which AB and AD are adjacent sides. On AB, AP is cut off equal to one-third of AB, and on AD, AQ is cut off equal to one-third of AD; show that the straight lines joining CP and CQ divide the parallelogram into three equal parts. (12)

11. Show how to construct a rectangle equal to a given rectilineal figure having five sides. (13)

12. ABC is an isosceles triangle, whose equal sides are AB and AC; produce AB to D, so that BD may equal BC, and join DC. Show that the angle ACD is three times the angle ADC. (13)

1881.

7. If three straight lines terminated at a point are such that no one of them falls within the angle made by the other two, show that the three angles just formed are together equal to four right angles. (8)

8. Prove that the opposite sides of a parallelogram are equal to one another and the diameter divides it into two equal parts. (9)

9. Show how to describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilinear angle, and prove your construction. (10)

10. In a right-angled triangle, show that the distance of the middle point of the hypotenuse from the right angle is equal to half the hypotenuse. (12)

11. ABC is an isosceles triangle, whose vertex is A, and ACD is an

* This number is given as in the actual examination paper, but it is clearly a misprint for (6).

isosceles triangle, whose vertex is C. Show that, if B, C, and D are in one right line, the angle at B is double the angle at D. (13)

12. Show that the sum of the perpendiculars let fall from any point within a parallelogram upon the four sides is the same whatever be the point selected. (13)

1882.

7. Prove that the diagonals of a parallelogram bisect one another, and that, if they intersect at right angles, the parallelogram must be either a square or a rhombus. (10)

8. If a side of a triangle be produced, the exterior angle thus formed is equal to the two interior and opposite angles.

The sides AB, BC, CA of a triangle are produced so as to form three exterior angles; if two of these exterior angles are together double of the third, show that one of the angles of the triangle is two-thirds of a right angle. (10)

9. Let ABC be an isosceles triangle, of which A is the vertex. Draw CF meeting AB produced in F, so that $CF=AC$, and draw BG meeting AC produced in G, so that $BG=AB$; let BG and CF meet in O. Show that the angle BOC is three times the angle at A. (10)

10. Construct the parallelogram whose diagonals and one side are equal to three given straight lines.

What condition must be fulfilled by the given lines that it may be possible to construct the parallelogram? (12)

11. Prove that the complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.

If straight lines be drawn from any point in the diagonal of a parallelogram to the two corners through which the diagonal does not pass, they form two pairs of equal triangles. (14)

12. Show that the straight line which joins the middle points of two sides of a triangle divides the triangle into parts, of which one is three times as large as the other. (14)

1883.

7. Draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be drawn at right angles to CD and meet it in B; join AC, AD; if BC is greater than BD, show that AC is greater than AD. (8)

8. In any triangle (ABC), show that, if the angle A is greater than the angle B, the side BC will be greater than the side CA.

Let ABC be an equilateral triangle, and let a point Q be taken in BC produced (in the direction B to C); show that Q is nearer to A than it is to B. (14)

9. If a straight line falling on two other straight lines makes the interior angles on the same side together equal to two right angles, the two straight lines are parallel to each other.

If A, B, C, D are the angular points of a quadrilateral taken in order, and if the angles at A and B are together equal to those at C and D, show that two sides of the quadrilateral are parallel to one another. (10)

10. Equal triangles on the same base and on the same side of it are between the same parallels.

If a quadrilateral is divided into four equal triangles by its diagonals, show that it is a parallelogram. (12)

11. Draw a triangle ABC, and through A draw a line parallel to BC; show how to draw through B a line cutting AC in P and the above-mentioned parallel in Q, so that BP shall be one-third of PQ. (14)

12. If one diagonal (AC) of a quadrilateral bisects the other diagonal (BD), show that AC also bisects the quadrilateral itself. (10)

1884.

7. Two angles of a triangle are equal; show that the sides opposite to these angles are equal.

If the equal sides of an isosceles triangle are produced, and the exterior angles between the base and the sides produced are bisected; show that the triangle contained by the base and the bisectors is isosceles. (8)

8. Show how to construct an equilateral triangle, having given three lines severally equal to the distances from its angular points to a point within it.

State the conditions that must be satisfied by the three given lines in order that the construction may be possible. (16)

9. If two straight lines cut one another, show that the vertically opposite angles are equal to one another.

On the sides of an equilateral triangle three equilateral triangles are drawn external to the given triangle; show that the whole figure is an equilateral triangle. (10)

10. Show how to draw a straight line through a given point parallel to a given straight line.

Draw a line DE parallel to the base BC of a triangle ABC, cutting AB in D and AC in E, so that DE shall be equal to the sum of BD and CE. (14)

11. Show that parallelograms on the same base and between the same parallels are equal.

Show how to construct an isosceles triangle equal to a given triangle. (10)

12. ABC is a triangle right-angled at C; draw CD at right angles to AB; in CD take CE equal to BD, and in CA take CF equal to AD; join AE and BF; and show that AE and BF are equal. (12)

APPENDIX.

ALTERNATIVE DEMONSTRATIONS.

Alternative proofs which are interesting for one reason or another are numerous. A short series of them is here introduced, comprising those only which have a distinct advantage over the ordinary proofs both in point of simplicity and of brevity, and which also fulfil the conditions of any of the ordinary examinations.

1. Symbols and Abbreviations.

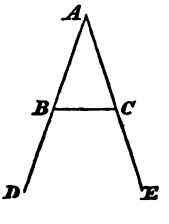
$\therefore$ therefore;	$\because$ because;	$=$ is *or* are equal to;
$\angle$ angle;	$\parallel$ parallel;	$\perp$ perpendicular to, *or* at right angles to;
$\triangle$ triangle;	sq. square;	$\square^{m}$ parallelogram;
pt. point;	$\odot$ circle;	$\odot^{ce}$ circumference;
str. straight;	rt. right;	equil. equilateral;
adj. adjacent;	opp. opposite;	int^{r}. interior;
ext^{r}. exterior;	$<$ less than;	$>$ greater than.

2. Proposition V.—Theorem.

The angles at the base of an isosceles triangle are equal to one another, and, if the equal sides be produced, the angles on the other side of the base are equal.

Let the $\triangle ABC$ be isosceles, having AB, AC equal.
And let AB, AC be produced to D and E.

Then $\angle ABC = \angle ACB$, *and* $\angle DBC = \angle ECB$.



For, if the complete figure were taken up, reversed, and then placed with
A in its former position,
and AD along the former position of AE;
then, $\because$ $\angle DAE$ and $\angle EAD$ are the same,
AE would fall along the former position of AD.
Also, $\because AB = AC$,
B would fall where C was,
and C would fall where B was;
$\therefore$ *BC would fall where CB was.* (Ax. 10)
Hence $\angle ABC$ would occupy the former position of $\angle ACB$, and $\angle DBC$ the former position of $\angle ECB$.
$\because \angle ABC = \angle ACB$ and $\angle DBC = \angle ECB$. (Ax. 8)
Q. E. D.

Note 1.—A proof similar in nature to the above is easily constructed for Prop. VI.; but no saving of space is thereby effected.

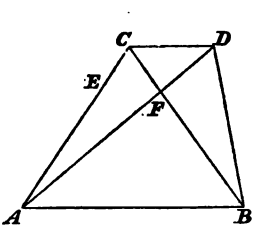
Note 2.—The first part of Prop. V. may be proved as a particular case of Prop. IV.

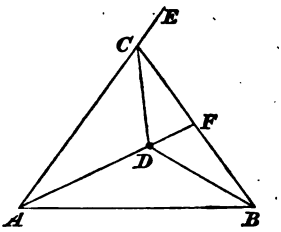
For BA, AC, and $\angle BAC$
$= CA$, AB, and $\angle CAB$, *each to each*,
in the $\triangle ABC$, and the same $\triangle ACB$.
$\therefore$ the $\angle$s opposite BA and CA are equal. (I. 4)

3. Proposition VII.—Theorem.

Upon the same base and upon the same side of it there cannot be two triangles which have not only their sides which are terminated in one extremity of the base equal to one another, but also those which are terminated in the other extremity.

If it be possible, upon the same base, and on the same side of it, let there be two $\triangle$s ACB, ADB,
which have not only CA, DA terminated in A equal,
but also CB, DB terminated in B.
When the vertex of each $\triangle$ lies without the other $\triangle$,
take E in AC and F where AD cuts BC;
when the vertex of one $\triangle$ lies within the other,
take E in AC produced and F where AD produced cuts BC.





Then, $\because AC = AD$,

$\therefore$ the $\angle FDC$ = the $\angle ECD$ (I. 5)

$\therefore$ *the* $\angle FDC$ *is* $>$ *the* $\angle FCD$.

Again, $\because BC = BD$,

$\therefore$ the $\angle BDC$ = the $\angle BCD$, (I. 5)

$\therefore$ *the* $\angle FDC$ *is* $<$ *the* $\angle FCD$;

so that the $\angle FDC$ is both $>$ and $<$ the $\angle FCD$;

which is absurd.

The case in which the vertex of one $\triangle$ lies upon a side of the other needs no proof.

Wherefore, &c. Q. E. D.

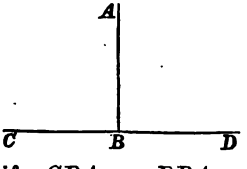
Note.—The proof just given will be seen to apply at the same time to both the cases usually given.

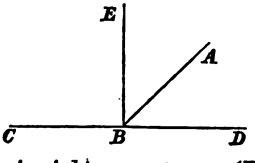
4. Proposition XIII.—Theorem.

Any two adjacent angles which one straight line makes with another, are together equal to two right angles.

Let BA make with CD the adj. $\angle$s CBA, DBA.

Then CBA *and* DBA *shall together* = *two rt.* $\angle$*s.*





For, if $\angle CBA = \angle DBA$, *each is right.* (Def. 10)

But, if CBA does not = DBA,

draw $BE \perp CD$, *so that* CBE, EBD *are rt.* $\angle$*s.* (I. 11)

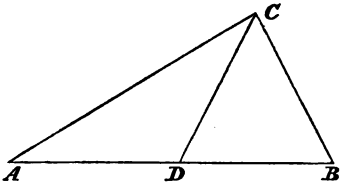
Then $\angle DBA$ and a part of $\angle ABC$ together make up the rt. $\angle DBE$,
and the remaining part of ABC forms the other rt. $\angle EBC$.
Therefore the two $\angle$s DBA, ABC together = two rt. $\angle$s.
Q. E. D.

Note.—A considerable simplification may be introduced into the usual proof, as already given in the text, by merely denoting the three small angles of the second figure by α, β, γ. The same expedient may be used with advantage in XIV., XV., XXVIII., XXIX.

5. Proposition XX.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be any Δ;
then shall AC, CB *be* $> AB$; CB, BA *be* $> AC$; *and* BA, $AC > BC$.
Bisect the $\angle ACB$ by the str. line CD, meeting AB in D, (I. 9)



Then $\angle ACD = \angle BCD$; but $\angle ADC$ is $> BCD$. (I. 16)
$\therefore$ $\angle ADC$ is $> \angle ACD$;
$\therefore$ *side* AC *is* $> AD$ (I. 19)
Similarly, *side* BC *is* $> BD$.
$\therefore$ the two sides AC, CB together are $> AB$.
In the same way we may prove the other inequalities.
Hence, any two sides, &c. Q. E. D.

6. Proposition XXIV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the contained angle of the one greater

than the contained angle of the other, then the base of that which has the greater angle shall be greater than the base of the other.

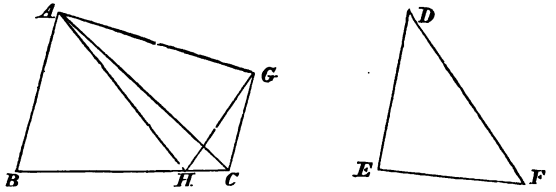
In the $\triangle$s ABC, DEF, let $BA, AC = ED, DF$, each to each, but let the $\angle BAC$ be $> \angle EDF$.

Then shall the base BC be $>$ the base EF.

Make $\angle CAG = \angle FDE$ (I. 23), $AG = DE$; and join CG.

Then $CA, AG = FD, DE$, each to each; and $\angle CAG = FDE$;

$\therefore$ *the base CG = base EF.* (I. 4)



Bisect $\angle BAG$. (I. 9)

$\because$ $\angle BAC$ is $>$ half of $\angle BAG$,

the bisecting line will meet BC.

Let it meet BC in H; and join GH.

Then $BA, AH = GA, AH$, each to each, and $\angle BAH = \angle GAH$;

$\therefore$ the base BH = base GH,

$\therefore$ *the base BC = GH, HC together;*

But GH, HC together are $> GC$. (I. 20)

$\therefore$ BC is $> GC$; $\therefore$ BC is $> EF$. Q. E. D.

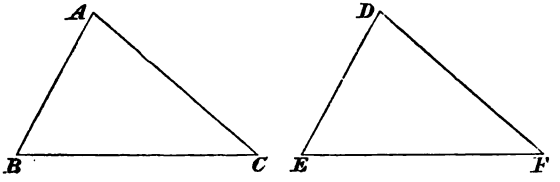
7. Proposition XXVI.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles or the sides opposite to either pair of equal angles; then shall their other sides be equal, each to each, and also their third angles.

First, let the $\triangle$s ABC, DEF have

$\angle B = \angle E$, $\angle C = \angle F$, and side BC = side EF.

Then shall $AB = DE$, $AC = DF$, and $\angle A = \angle D$.

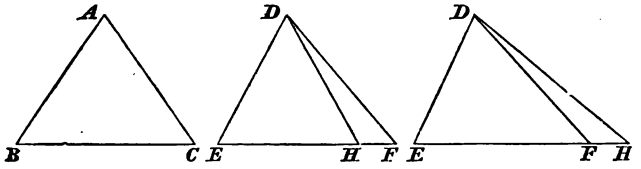


Apply ΔABC to ΔDEF, putting B on E and BC on EF.
Then C will fall on F, $\because BC = EF$;
Also BA will fall on ED, $\because \angle B = \angle E$;
and CA will fall on FD, $\because \angle C = \angle F$.

$\therefore$ *A will fall on D;*

$\therefore$ BA will coincide with ED and be equal to it, (Ax. 8)
CA will coincide with FD and be equal to it,
and $\angle A$ will coincide with $\angle D$ and be equal to it.

Secondly, let the Δs ABC, DEF have
$\angle B = \angle E$, $\angle C = \angle DFE$, and side AB = side DE.
In this case, also, they shall be equal in all respects.



Apply ΔABC to ΔDEF, putting A on D and AB on DE.
Then B will fall on E, $\because AB = DE$;
also BC will fall on EF, $\because \angle B = \angle E$;
also AC will fall on DF.

For, if not, let it meet EF or EF produced in H.
Then, in either case,

$\because \angle C = \angle DFE$; $\therefore \angle DHE = \angle DFE$;

which is impossible. (I. 16)

$\therefore$ *AC does fall on DF;*

and, as before, the Δs ABC, DEF coincide;
and are equal in all respects. (Ax. 8)
Wherefore, &c. Q. E. D.

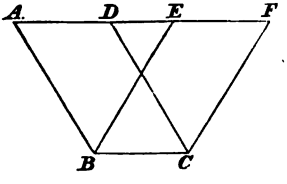
8. Proposition XXXV.—Theorem.

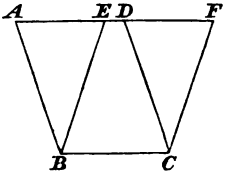
Parallelograms upon the same base and between the same parallels are equal.

Let $ABCD$, $EBCF$ be the two $\square^{ms}$ upon the same base BC, and between the same $\|^s$.

Then shall $\square^m ABCD = \square^m EBCF$.

First, let AD, EF be not terminated in the same pt.





then, $\because AB, DC$ are $\|^s, \therefore \angle FDC = \angle A$;

and $\because EB, CF$ are $\|^s, \therefore \angle F = \angle AEB$;

$\therefore$ remaining $\angle DCF =$ remaining $\angle ABE$. (I. 32)

Also $DC = AB$, and $CF = BE$; (I. 34)

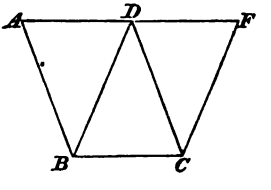
$\therefore \triangle DCF = \triangle ABE$. (I. 4)

Take $\triangle DCF$ from trapezium $ABCF$; and then replace it: afterwards take $\triangle ABE$ from the same.

The two remainders will be equal. (Ax. 3)

$\therefore \square^m ABCD = \square^m EBCF$.

Secondly, let AD, EF be terminated in one pt., E coinciding with D; so that the $\square^{ms}$ become $ABCD$, $DBCF$.



It is plain that these are equal;

for each is double of $\triangle DBC$. (I. 34)

Therefore, parallelograms, &c. Q. E. D.

9. PROPOSITION XLVII.—THEOREM.

In a right-angled triangle the square upon the hypotenuse is equal to the sum of the squares upon the other sides.

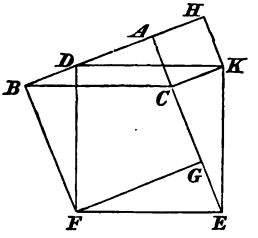
Let the ΔABC be rt.-$\angle^d$, having the $\angle A$ rt.,

then the sq. on BC = *sum of sqq. on* BA, AC.

On AB describe the sq. $ABFG$; and on AC the sq. ACE (I. 46

Produce AG; and make BD, GE each = AC.

Join DF, FE, EK, KD.



Now it is easy to shew that the four Δs DHK, KCE, EG and FBD each = ΔBAC in all respects, (I.

$\therefore$ they = each other in all respects,

and have the sides subtending their right angles all equal;

$\therefore$ the figure $DFEK$ is equilateral.

Now, by the same proof, $\angle CEK = \angle GFE$,

$\therefore$ $\angle FEK$ = a rt. $\angle$.

Similarly the other $\angle$*s of* $DFEK$ *are rt.*

$\therefore$ $DFEK$ is a sq.

Now we may superpose DHK on ECK, and DBF on FGE and the remaining part of the two sqq. BG, CH coincides with the remainder of the sq. DE;

$\therefore$ *sq.* DE = *sum of sqq.* BG, CH.

But $DK = BC$;

$\therefore$ sq. on BC = sum of sqq. on BA, AC.

Note.— The foregoing shews how we may dissect the two smaller squares so as to fill up with the pieces the area of the larger one. For this reason chiefly it is inserted; as, fully stated, it is probably at least as long as the usual proof.

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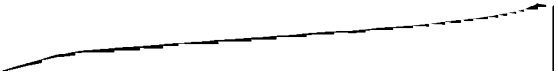


Diagram II.

FIG. 4. All the sides of the small triangles

equal.

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